THEORY OF PROBABILITY

Since uncertainty is an integral part of human life, people have always been interested - consciously or unconsciously - in evaluating probabilities. Having its origin associated with gamblers, the theory of probability today is an indispensable tool in the analysis of situations involving uncertainty. It forms the basis for inferential statistics as well as for other fields that require quantitative assessments of chance occurrences, such as quality control, management, decision analysis and almost all areas in Physics, Biology, Engineering and Economics or Social life.

Probability

Definition:-A probability is a quantitative measure of uncertainty

OR

The theory of probability is an attempt to measure the degree of uncertainty in the results of such experiments.

OR

A number that conveys the strength of our belief in the occurrence of an uncertain event. Probability, in common dialogue, refers to the chance of occurrence of an event or happening. In order that we are able to compute it, a proper understanding of certain basic concepts in probability theory is required. These concepts are an experiment, a sample space, and an event.

Experiment:-

The term experiment is used in probability theory in a much broader sense than in physics or chemistry. Any action, whether it is the drawing a card out of a deck of 52 cards, or reading the temperature, or measurement of a product's dimension to ascertain quality, or the launching of a new product in the market, constitute an experiment in the probability theory.

The experiments in probability theory have three things in common:

[i] There are two or more outcomes of each experiment.

[ii] It is possible to specify the outcomes in advance.

[iii] There is uncertainty about the outcomes

Outcome Definition:-

An **outcome** of an experiment is some observation or measurement.

Experiment Definition:-

An **experiment** is a process that leads to one of several possible **outcomes.**

Experiments can be of two types.

[1] Deterministic and [2] Non-deterministic (random)

Deterministic Experiment:-

A single outcome of an experiment is called deterministic experiment.

For example

[i] If a ball is thrown in the air it is sure that it will fall down.

[ii] If temperature of water is increased to $100^{o}C$ then it is sure that it starts boiling.

Non-deterministic Experiment:-

There are more than one possible outcomes of an experiment is called random experiment. For example:

[i] If a coin is tossed repeatedly until a head is obtained.

[ii] In case of a new born baby, the sex cannot be predicted with certainty.

[iii] Drawing a card from a pack of playing cards is a random experiment.

[iv] Throwing a die is a random experiment.

Sample Space:-

The **sample space** is the set of all possible outcomes of an experiment. It is denoted by S or Ω.

For example

[i] If a coin is tossed repeatedly until a head is obtained.

Ω = {T,TH,TTH,TTTH,….}

[ii] Sex of a new born baby, the outcomes are.

Ω = {M, F}

[iii] Launching of a New Product

Ω = {Success, Failure}

[iv] If a coin is tossed, the sample space will be as follows

Ω = {H, T}

[v] Similarly if two coins are tossed the following sample space is generated:

Ω = {(H, H), (H, T), (T, H), (T, T)}

[vi] When 3 coins are thrown simultaneously, the sample space S is given by.

Ω = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

There are two types of sample spaces:

[1] Finite and [2] Countable Infinite

[1] Finite sample space:-

If the number of outcomes of a random experiment is finite then sample space of such experiment is called finite sample space.

[i] Launching of a New Product

Ω = {Success, Failure}

[ii] If two coins are tossed simultaneously then the outcomes are

Ω = {(H,H), (H,T), (T,H), (T,T)}

[iii] A pair of dice is rolled then the outcomes are

Ω = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),

(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),

(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),

(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)

(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),

(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

[2] Countable Infinite sample space:-

If the number of outcomes of a random experiment is countable infinite then sample space of such experiment is called countable infinite sample space.

[i] A student appears for an examination till he passes. Then the sample sapace is

Ω = {Success, Failure Success, Failure Failure Success, Failure Failure Failure Success,…}

[ii] Number of students arriving at the cash counter in a college then the outcomes is

Ω = {0,1,2,….}

EVENT:

An event, in probability theory, constitutes one or more possible outcomes of an experiment.

Definition: The results of an experiment are known as events

OR

An **event** is a subset of a sample space.

It is a set of basic outcomes. We say that the event occurs if the experiment gives rise to a basic outcome belonging to the event.

[i] For the experiment of drawing a card, we may obtain different events A, B, and C like:

A: The event that card drawn is king of club

B: The event that card drawn is red

C: The event that card drawn is ace

In the first case, out of the 52 sample points that constitute the sample space, only one sample point or outcome defines the event, whereas the number of outcomes used in the second and third case is 13 and 4 respectively.

[ii] If a coin is tossed head (H) and tail (T) are two different events.

[iii] 1,2,3,4,5,6 are different events when a die is thrown.

Elementary event:-

Definition: A single outcome of an experiment is called a **basic outcome** or an **elementary event**.

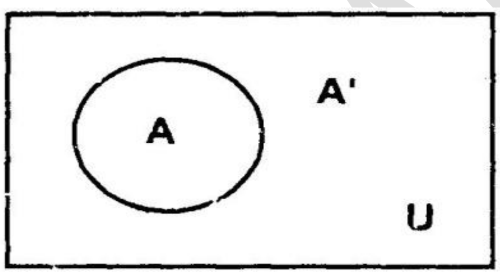
Any particular card drawn from a deck is a basic outcome.

Impossible event:-

Definition: An event which does not contain any sample point is called an impossible event.

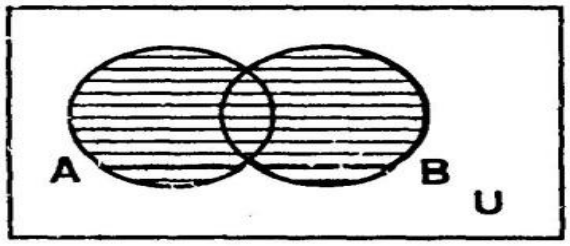
When an event cannot occur (impossible event), its probability is zero. The probability of the empty set is zero i.e. P(Φ) = 0. In a deck where half the cards are red and half are black, the probability of drawing a green card is zero because the set corresponding to that event is the empty set: there are no green cards

Complementary event:



The complement of an event A is the aggregate of all the sample points of sample space S which do not belong to A. It is denoted be A' or $_{A}$ .e.g. If A is an event of getting an odd number when a die is thrown, then event of not getting an odd number i.e. getting an even number is the complement of event A and it is denoted 'by A' or $_{A}$

Union of two events:

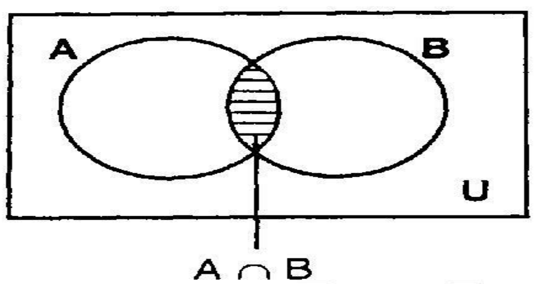


A ∪ B

The union of two events A and B is denoted by A B. It is the aggregate of all sample points belonging to either A or B or both.

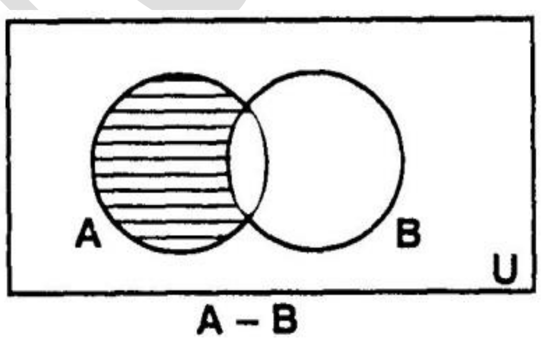
e.g. If A is an event that a student plays cricket and B is an event that a student plays hockey then A B. represents an event that a student plays either cricket or hockey or both.

Intersection of two events:



The intersection two events A and B is denoted by A B. It is the aggregate of all sample points belonging to A and B both. When two events A and B occur simultaneously we say that A B has occurred e.g., If A is an event that a student plays cricket and B is an event that a student plays hockey, then A B represents an event that a student plays cricket and hockey both.

Difference Event:

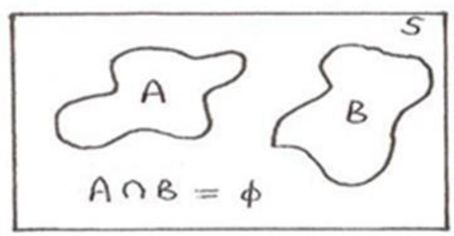


The difference of two events A and B is the event that A happens and B does not happen. It is denoted by A - B.

A - B = A   B’

Mutually Exclusive Events

When the sets corresponding to two events are disjoint (i.e., have no intersection), the two events are called **mutually exclusive**.



For mutually exclusive events, the probability of the intersection of the events is zero. This is so because the intersection of the events is the empty set, and we know that the probability of the empty set is zero.

For mutually exclusive events A and B: $P(A \cap B) = 0$

Exhaustive Events:-The total number of possible outcomes of a random experiment are known as exhaustive events i.e. the events $A_1, A_2, A_3, \ldots, A_n$ defined on S are said to be exhaustive if $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S$

For example, if $S = \{0,1,2,3,4,5,6,7,8,9\}$

If $A_1 = \{0,2,4,6,8\}$, $A_2 = \{3,5\}$, $A_3 = \{1,7,8\}$, $A_4 = \{9\}$ then

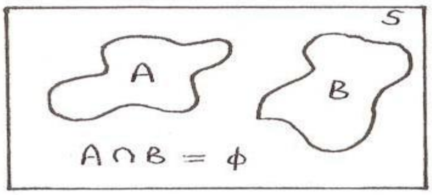
$A_1 \cup A_2 \cup A_3 \cup A_4 = S$

$\therefore A_1, A_2, A_3$ and A_4 are exhaustive.

Equally Likely Events: The events A and B are said to be equally likely if none of them is expected to occur in preference to other i.e. chances of happening them are the same.

For example, in tossing an unbiased coin, the events ‘getting head’ and ‘getting tail’ are equally likely events.

Operations on Events:

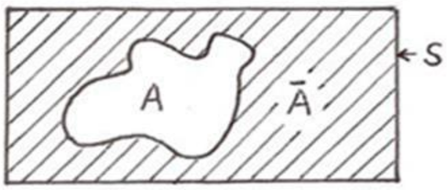


Complement Rule:-

The **Rule of Complements** defines the probability of the complement of an event in terms of the probability of the original event.

Consider event A defined over the sample space S. The complement of set A, denoted by $\bar{A}$, is a subset, which contains all outcomes, which do not belong to A

For Example: If the probability of rain tomorrow is 0.3, then the probability of no rain tomorrow must be 1 - 0.3 = 0.7.



Relative Complement:

Let A and B are two events defined on sample space S. The relative complement of A with respect to B is denoted by $(\bar{A} \cap B)$. It means that contains all points of B but not of A.

Also, relative complement of B with respect to A is denoted by $(A \cap \bar{B})$. It means that contains all points of A but not of B.

Approaches to Probability Theory

Three different approaches to the definition and interpretation of probability have evolved, mainly to cater to the three different types of situations under which probability measures are normally required. We will study these approaches with the help of examples of distinct types of experiments.

Consider the following situations marked by three distinct types of experiments. The events that we are interested in, within these experiments, are also given.

Situation-I		Situation-II		Situation-III	
Experiment	**Event A**	**Experiment**	**Event A**	**Experiment**	**Event A**
Drawing a Card Out of a Deck of 52 Cards	On any draw, a king is there	Administering a Taste Test for a New Soup	A consumer likes the taste	Commissioning a Solar Power Plant	The plant turns out to be a successful venture

Situation-I: The Classical Approach

The first situation is characterized by the fact that for a given experiment we have a sample space with equally likely basic outcomes. When a card is drawn out of a well-shuffled deck, every one of the cards (the basic outcomes) is as likely to occur as any other. This type of situations, marked by the presence of **"equally likely"** outcomes, gave rise to the ***Classical Approach*** to the probability theory. In the Classical Approach, probability of an event is defined as the **relative size** of the event with respect to the size of the sample space. Since there are 4 kings and there are 52 cards, the size of A is 4 and the size of the sample space is 52. Therefore, the probability of A is equal to 4/52.

The rule we use in computing probabilities, **assuming equal likelihood of all basic outcomes, is as follows:**

Probability of the event A is denoted by P(A) and is given by

$$P(A)=\frac{n(A)}{n(S)}$$

$n(A)$= the number of outcomes favourable to the event A

$n(S)$= total number of outcomes

Situation II: The Relative Frequency Approach

If we try to apply the classical definition of probability in the second experiment, we find that we cannot say that consumers will equally like the taste of the soup. Moreover, we do not know as to how many persons have been tested. This implies that we should have the past data on people who were administered the soup and the number that liked the taste. In the absence of past data, we have to undertake an experiment, where we administer the taste test on a group of people to check its effect.

The **Relative Frequency Approach** is used to compute probability in such cases. As per this approach, the probability of occurrence of an event is given by the observed relative frequency of an event in a very large number of trials. In other words, the probability of occurrence of an event is the ratio of the number of times the event occurs to the total number of trials. The probability of the event B:

$$P(B)=\frac{n}{N}$$

n = the number of times the event occurs

N = total number of trials

It is appreciated in this approach that, in order to take such a measure, we should have the soup tested for a large number of people. In other words, the total number of trials in the experiment should be very large.

Situation-III: The Subjective Approach

The third situation seems apparently similar to the second one. We may be tempted here to apply the Relative Frequency Approach. We may calculate the probability of the event that the venture is a success as the ratio of number of successful ventures to the total number of such ventures undertaken i.e. the relative frequency of successes will be a measure of the probability. However, the calculation here presupposes that either

(a) It is possible to do an experiment with such ventures, or

(b) That past data on such ventures will be available

In practice, a solar power plant being a relatively new development involving the latest technology, past experiences are not available. Experimentation is also ruled out because of high cost and time involved, unlike the taste testing situation. In such cases, the only way out is the **Subjective Approach** to probability. In this approach, we try to assess the probability from our own experiences. We may bring in any information to assess this. In the situation cited, we may, perhaps, look into the performance of the commissioning authority in other new and related technologies Therefore the Subjective Approach involves personal judgment, information, intuition, and other subjective evaluation criteria. A physician assessing the probability of a patient's recovery and an expert assessing the probability of success of a merger offer are both making a personal judgment based upon what they know and feel about the situation. The area of subjective probability - which is relatively new, having been first developed in the 1930*s* - is somewhat controversial. One person's subjective probability may very well be different from another person's subjective probability of the same event. We may note here that since the assessment is a purely subjective one, it will vary from person to person and, therefore, subjective probability is also called **Personal Probability.**

Three Approaches – A Comparative View

As already noted, the different approaches have evolved to cater to different kinds of situations. So these approaches are not contradictory to one another. In fact, these complement each other in the sense that where one fails, the other becomes applicable. These are identical inasmuch as probability is defined as a ratio or a weight assigned to the occurrence of an event. However, in contrast to the Subjective measure of the third approach, the first two approaches - Classical and Relative Frequency - provide an objective measure of probability in the sense that no personal judgment is involved.

We can bring out the commonality between the Classical Approach and the Relative Frequency Approach with the help of an example. Let us assume that we are interested in finding out the chances of getting a head in the toss of a coin. By now, you would have come up with the answer by the Classical Approach, using the argument, that there are two outcomes, heads and tails, which are equally likely. Hence, given that a head can occur only once, the probability is $\frac{1}{2}$

Consider the following alternative line of argument, where the probability can be estimated using the Relative Frequency Approach. If we toss the coin for a sufficiently large number of times and note down the number of times the head occurs, the proportion of times that a head occurs will give us the required probability.

Axiomatic approach to probability:-

Let Ω be a sample space of a random experiment. Let A be any event defined on Ω and P is called the probability function or probability measure, which satisfied the following axioms.

Axioms 1: The probability of an event A, written as P(A), must be a number between zero and one, both values inclusive. i.e. $0 \le P(A) \le 1$ for any event A on S.

Axioms 2: The probability of occurrence of one or the other of all possible events is equal to one. As S denotes the sample space or the set of all possible events, we write $P(S) = 1$. Thus in tossing a coin once; P(a head or a tail) = 1.

Axioms 3: If two events are such that occurrence of one implies that the other cannot occur, then the probability that either one or the other will occur is equal to the sum of their individual probabilities. Thus, in a coin-tossing situation, the occurrence of a head rules out the possibility of occurrence of tail. These events are called **mutually exclusive events**. In such cases then, if A and B are the two events respectively, then

P (A or B) = P (A) + P (B)

i.e. P(Head or Tail) = P (Head) + P (Tail)

P(Head $\cup$ Tail) = P (Head) + P (Tail)

$P(A \cup B) = P(A) + P(B)$, for every pair of mutually exclusive events defined on Ω

Interpretation of Probability:

Within the range of values 0 to 1, the greater the probability, the more confidence we have in the occurrence of the event in question.

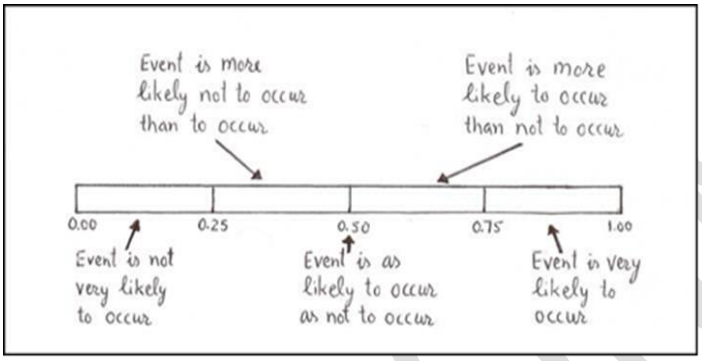
[i] A probability of 0.95 implies a very high confidence in the occurrence of the event.

[ii] A probability of 0.80 implies a high confidence.

[iii] When the probability is 0.5, the event is as likely to occur as it is not to occur.

[iv] When the probability is 0.2, the event is not very likely to occur.

[v] When we assign a probability of 0.05, we believe the event is unlikely to occur, and so on.



Construct the sample space for each of the following experiment:

[i] A coin is tossed 4 times.

[ii] A lot of six items contains two defective items. Items are drawn one after another without replacement until both the defectives have been found. The number of draws required is recorded.

[iii] For a sociological study 3 out of 5 cities A,B,C,D,E are selected.

[iv] Total quantities sold of a particular product on any way.

[v] A die is tossed until it shows 6 for the first time

Solution: A coin is tossed 4 times then the sample space is

S = {HHHH, HHHT, HHTH, HTHH…}

Formulae for Permutation and combination:

[1] A permutation of n objects taken r at a time is an ordered arrangement of r objects from n objects the number of distinct permutation of n things taken r at a time is denoted by ${}^{n}p_{r}$ is given by

(i) ${}^{n}p_{r} = \frac{n!}{(n-r)!}$ (ii) ${}^{n}p_{n} = \frac{n!}{0!}$ (iii) ${}^{n}p_{0} = \frac{n!}{(n-0)!}$ (iv) ${}^{n}p_{1} = \frac{n!}{(n-1)!}$

[2] Total number of different permutation of n distinct objects taken r at a time with repetition of any number of times is ${}^{n}p_{r} = n^{r}$

[3] The number of permutation of n objects are alike one type, n_2 are alike of second type and so on with n_k objects alike if k^{th} type is given by

$$\frac{n!}{n_1!,n_2!,\dots,n_k!}$$

[4] Combination of n different objects are taken r at a time is a selection of any r objects out of n objects without considering order in the selection the total number of different combination of n distinct taken r at a time without repetition is denoted by ${}^{n}c_{r}$

(i) ${}^{n}c_{r} = \frac{n!}{r!(n-r)!}$ (ii) ${}^{n}c_{n} = 1 = {}^{n}c_{0}$

(iii) ${}^{n}c_{r} = {}^{n}c_{n-r} = \frac{n!}{n!(n-n+r)!} = \frac{n!}{r!(n-r)!}$ (iv) ${}^{n}c_{1} = 1$

[5] $2^{n} = (1+n)^{n} = {}^{n}c_{0} + {}^{n}c_{1} + {}^{n}c_{2} + \dots + {}^{n}c_{n}$

[6] Sum of 1^{st} n natural number $= \frac{n(n+1)}{2} = \sum_{c=1}^{n} i$

[7] Sum of squares of 1^{st} n natural number $= \sum_{i=1}^{n} i^{2} = \frac{n(n+1)(2n+1)}{6}$

[8] Sum of cubes of 1^{st} n natural number $= \sum_{c=1}^{n} i^{3} = \frac{n^{2}(n+1)^{2}}{4} = \left(\frac{n(n+1)}{2}\right)^{2}$

[9] or = + [10] and = ×

Theorems on Probability

[1] Probability of the complementary event A of A is given by
$P(\bar{A})=1-P(A)$ or $P(\bar{A})+P(A)=1$

Proof: From Venn diagram A and $\bar{A}$ are mutually exclusive event, so that

$$A \cup \bar{A} = S \qquad \cdots(i)$$

If we take probability of (i) then, we get

$$P(A \cup \bar{A}) = P(S)$$
$$P(A)+P(\bar{A}) = P(S) \qquad (\because \text{axioms -3})$$
$$P(A)+P(\bar{A}) = 1 \qquad (\because \text{axioms 2})$$
$$\text{Hence, } P(\bar{A}) = 1-P(A)$$

[2] For an event A defined on Ω then prove that $0 \le P(A) \le 1$

Proof: We know that

$$P(\bar{A}) = 1-P(A) \qquad ...(i)$$
$$P(A) \ge 0 \qquad \cdots(\because \text{axioms-1})$$

$$1-P(A) \ge 0 \qquad ...(\because (i))$$
$$1 \ge P(A)$$
$$\Rightarrow P(A) \le 1 \qquad ...(ii)$$

Similarly by axioms (1)

$$P(A) \ge 0$$
$$\text{or } 0 \le P(A) \qquad ...(iii)$$
$$\text{from (ii) and (iii), we get}$$
$$0 \le P(A) \le 1$$

[3] The probability of an impossible event is zero i.e. $P(\phi)=0$

Proof: ϕ and S, are complement to each other

$(\Omega \cup \phi) = \Omega$

$P(\Omega \cup \phi) = \Omega$

$P(\phi) + P(S) = 1 \qquad \cdots(\because \text{axioms-3})$

$P(\phi) + 1 = 1 \qquad (\because P(S) = 1)$

$P(\phi) = 0$

[4] For any two events A and B, prove that $P(\bar{A} \cap B) = P(B) - P(A \cap B)$.

Proof: From the Venn diagram, we can write

$B = (A \cap B) \cup (\bar{A} \cap B) \quad ....(i)$

Where, $A \cap B$ and $\bar{A} \cap B$ are disjoint event. Hence, by axioms-3, we get

$P(B) = P\left[(A \cap B) \cup (\bar{A} \cap B)\right]$

$\quad = P(A \cap B) + P(\bar{A} \cap B)$

$P(\bar{A} \cap B) = P(B) - P(A \cap B)$

Hence, proved

[5] For any two events A and B, prove that $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

Proof: From the Venn diagram, we can write

$A = (A \cap B) \cup (A \cap \bar{B}) \quad ....(i)$

Where, $A \cap B$ and $\bar{A} \cap B$ are disjoint event. Hence, by axioms-3, we get

$P(A) = P\left[(A \cap B) \cup (A \cap \bar{B})\right]$

$P(A) = P(A \cap B) + P(A \cap \bar{B}) \qquad \cdots(\because \text{axioms-3})$

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$

Hence, proved

[6] If $B \subset A$, then prove that $P(A \cap \bar{B}) = P(A) - P(B)$.

Proof: When $B \subset A$, here A and B are mutually exclusive events so that

$A = B \cup (A \cap \bar{B}) \; ......(i)$

$$P(A) = P[B \cup (A \cap \bar{B})]$$
$$P(A) = P(B) + P(A \cap \bar{B}) \text{.......(axioms 3)}$$
$$P(A \cap \bar{B}) = P(A) - P(B)$$

Hence, proved

[7] If $B \subset A$, then prove that $P(B) \leq P(A)$

Proof: We know $P(A \cap \bar{B}) \geq 0$ ....by axioms-1

We have theorem $P(A \cap \bar{B}) = P(A) - P(B)$

$$P(A) - P(B) \geq 0$$
$$P(A) \geq P(B)$$
$$P(B) \leq P(A)$$

Hence, proved

[8] Addition theorem of probability

If A and B are any two events (subset of sample space) and are not disjoint then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof: From the Venn diagram, we have

$$(A \cup B) = A \cup (\bar{A} \cap B) \qquad \text{....(i)}$$

Where A and $(\bar{A} \cap B)$ are mutually exclusive events

$$P(A \cup B) = P[A \cup (\bar{A} \cap B)]$$
$$= P(A) + P(\bar{A} \cap B) \qquad \text{.....}(\because \text{axiom-3})$$
$$P(A \cup B) = P(A) + P(\bar{A} \cap B) \qquad \text{....(ii)}$$

But we know the theorem for any two events

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Equation (ii) becomes

$P(A\cup B)=P(A)+P(B)-P(A\cap B)$

OR

from (ii),we get

$P(A\cup B)=P(A)+P(\bar{A}\cap B)+P(A\cap B)-P(A\cap B)$

$\quad\quad =P(A)+[P(\bar{A}\cap B)+P(A\cap B)]-P(A\cap B)$

$P(A\cup B)=P(A)+P(B)-P(A\cap B)$

Corollary: If the events A and B are mutually exclusive event then $A\cap B=\phi$

[9] If A, B and C are any three events defined on Ω then

$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$

Proof: Let

$B\cup C=D$

$P(A\cup B\cup C)=P(A\cup D)$

$P(A\cup B\cup C)=P(A)+P(D)-P(A\cap D)$

$P(A\cup B\cup C)=P(A)+P(B\cup C)-P[A\cap(B\cup C)]$

$P(A\cup B\cup C)=P(A)+[P(B)+P(C)-P(B\cap C)]-P[(A\cap B)\cup(A\cap C)]$

$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(B\cap C)-P[(E\cup F)]$

$E=A\cap B$ and $F=A\cap C$

$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(B\cap C)-[P(E)+P(F)-P(E\cap F)]$

$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(B\cap C)-P(E)-P(F)+P(E\cap F)$

$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(B\cap C)-P(A\cap B)-P(A\cap C)$

$\quad\quad +P[(A\cap B)\cap(A\cap C)]$

$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(B\cap C)-P(A\cap B)-P(A\cap\cap C)+P(A\cap B\cap C)$

$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(B\cap C)-P(A\cap B)-P(A\cap C)+P(A\cap B\cap C)$

[10] Boole's inequality

If A and B are any two events defined on Ω, then $P(A\cup B)\le P(A)+P(B)$

Proof: We know that by addition theorem for any two events

$P(A\cup B)=P(A)+P(B)-P(A\cap B)$ and $P(A\cap B)\ge 0$ by axiom first

$P(A \cup B) \leq P(A) + P(B)$

[11] If A_1, A_2, A_3 and A_n are n events defined on Ω, then $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$

Proof: We prove the theorem by the method of induction

$$P(A \cup B) \leq P(A) + P(B) \qquad \ldots(i)$$

i.e. Result is true for n =2

Let us assume that the result holds for n=m

$$P\left[\bigcup_{i=1}^{m} A_i\right] \leq \sum_{i=1}^{m} P(A_i) \qquad \ldots.(ii)$$

Now, to prove that the result is true for n = m+1

Consider,

$$\bigcup_{i=1}^{m+1} A_i = \left(\bigcup_{i=1}^{m} A_i\right) \cup A_{m+1} \qquad \ldots(iii)$$

Let $\bigcup_{i=1}^{m} A_i = B$

Taking the probability of (iii), we get

$$P\left[\bigcup_{i=1}^{m+1} A_i\right] = P\left[B \cup A_{m+1}\right]$$

$$P\left[\bigcup_{i=1}^{m+1} A_i\right] \leq P(B) + P(A_{m+1}) \qquad [\because (i)]$$

$$P\left[\bigcup_{i=1}^{m+1} A_i\right] \leq P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1})$$

$$P\left[\bigcup_{i=1}^{m+1} A_i\right] \leq \sum_{i=1}^{m} P(A_i) + P(A_{m+1})$$

$$P\left[\bigcup_{i=1}^{m+1} A_i\right] \leq \sum_{i=1}^{m+1} P(A_i)$$

Hence, by induction

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$$

[12] If A_1, A_2, A_3 and A_n are n events defined on Ω, then $P\left(\bigcap_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} P(A_i) - (n-1)$

Proof: We prove the theorem by the method of induction

$P(A \cup B) \le 1 \qquad (\because 0 \le P \le 1)$

Numerical Examples:

[1] A card is drawn from a well-shuffled pack of playing cards. Find the probability that the card drawn is either a club or a king.

Solution: Let A be the event that a club is drawn and B the event that a king is drawn. There are 13 club cards and therefore, $A = {}^{13}C_1 = 13$

A card is drawn from a well-shuffled pack of playing cards then the sample space is:

$S = {}^{52}C_1 = 52$; $P(A) = \frac{13}{52} = \frac{1}{4}$; $P(B) = \frac{4}{52} = \frac{1}{13}$

Probability that the card drawn is a club and a king is: $P(A \cap B) = \frac{1}{52}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cup B) = \frac{1}{4} + \frac{1}{13} - \frac{1}{52}$$
$$P(A \cup B) = \frac{13 + 4 - 1}{52}$$
$$P(A \cup B) = \frac{16}{52}$$
$$P(A \cup B) = \frac{4}{13}$$

[2] Suppose your chance of being offered a certain job is 0.45, your probability of getting another job is 0.55, and your probability of being offered both jobs is 0.30. What is the probability that you will be offered at least one of the two jobs?

Solution: Let A be the event that the first job is offered and B the event that the second job is offered. Then,

$P(A) = 0.45;\ P(A) = 0.55$ and $P(A \cap B) = 0.30$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = 0.45 + 0.55 - 0.30$

$P(A \cup B) = 0.70$

[3] A card is drawn from a well-shuffled pack of playing cards. Find the probability that the card drawn is either a king or a queen.

Solution: Let A be the event that a king is drawn and B the event that a queen is drawn. Since A and B are two mutually exclusive events, we have,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = \frac{4}{52} + \frac{4}{52} - 0$

$P(A \cup B) = \frac{8}{52}$

$P(A \cup B) = \frac{4}{13}$

[4] Arrange the following probabilities in ascending order.

$P(A \cup B),\ P(A),\ P(A) + P(B), P(A \cap B)$

Solution:-

$A \cap B \subset A$

$P(A \cap B) \le P(A)$ $\quad [\because \text{If } A \subset B \text{ then } P(A) \le P(B)]$

Further $A \subset A \cup B$

$P(A) \le P(A \cup B)$ $\quad [\because \text{If } A \subset B \text{ then } P(A) \le P(B)]$

Boole's inequality

$P(A \cup B) \le P(A) + P(B)$

$P(A \cap B) \le P(A) \le P(A \cup B) \le P(A) + P(B)$

[5] Let A and B be two events defined on a sample space **Ω**, such that

$P(A)=\frac{3}{4}$ and $P(B)=\frac{5}{8}$, show that $\frac{3}{8}\leq\ P(A\cap B)\leq\frac{5}{8}$

Solution: $(A\cap B)\subset(B)$

$$P(A\cap B)\leq P(B)$$
$$P(A\cap B)\leq\frac{5}{8}\qquad\ldots(i)$$

Boole's inequality

$$P(A\cap B)\geq P(A)+P(B)-1$$
$$P(A\cap B)\geq\frac{3}{4}+\frac{5}{8}-1$$
$$P(A\cap B)\geq\frac{6+5-8}{8}$$
$$P(A\cap B)\geq\frac{3}{8}\qquad\ldots(ii)$$

From (i) and (ii)

$$\frac{3}{8}\leq P(A\cap B)\leq\frac{5}{8}$$

[6] If P(A) = P(B) =1, show that $P(A\cup B)=P(A\cap B)=1$

Solution: We have given, P(A) = P(B) =1

Boole's inequality

$$P(A\cap B)\geq P(A)+P(B)-1$$
$$P(A\cap B)\geq 1+1-1$$
$$P(A\cap B)\geq 1\qquad\ldots(i)$$

If $(A\cap B)\subset(B)$ then

$$P(A\cap B)\leq P(B)$$

But, $P(B)=1$

$$\therefore P(A\cap B)\leq 1\qquad\ldots(ii)$$

From (i) and (ii)

$$P(A \cap B) = 1$$

By addition theorem of probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cup B) = 1 + 1 - 1$$
$$P(A \cup B) = 1$$

Hence, $P(A \cup B) = P(A \cap B) = 1$

[7] Suppose A, B and C are three events defined on Ω such that

$$P(A) = P(B) = P(C) = \frac{1}{4}, P(A \cap C) = \frac{1}{8}, P(A \cap B) = P(B \cap C) = 0$$

Calculate
$(i) P(A \cup B \cup C)$ $(ii) P(A \cup C)$ $(iii) P(A' \cap B' \cap C')$ $(iv) P(A \cap C')$

Solution: We have given

$$P(A) = P(B) = P(C) = \frac{1}{4}, P(A \cap C) = \frac{1}{8}, P(A \cap B) = P(B \cap C) = 0$$

(i) By addition theorem of probability for three events

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$$

$$P(A\cup B\cup C)=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-0-0-\frac{1}{8}+0 \qquad \begin{bmatrix}\because P(A\cap B)=P(B\cap C)=0\\ \therefore P(A\cap B\cap C)=0\end{bmatrix}$$

$$P(A\cup B\cup C)=\frac{2+2+2-1}{8}$$

$$P(A\cup B\cup C)=\frac{5}{8}$$

$$(ii)\,P(A\cup C)=P(A)+P(C)-P(A\cap C)$$

$$P(A\cup C)=\frac{1}{4}+\frac{1}{4}-\frac{1}{8}$$

$$P(A\cup C)=\frac{2+2-1}{8}$$

$$P(A\cup C)=\frac{3}{8}$$

$$(iii)\,P(A'\cap B'\cap C')=P(A\cup B\cup C)'=1-P(A\cup B\cup C)$$

$$P(A'\cap B'\cap C')=1-\frac{5}{8}$$

$$P(A'\cap B'\cap C')=\frac{8-5}{8}$$

$$P(A'\cap B'\cap C')=\frac{3}{8}$$

$$(iv)\,P(A\cap C')=P(A)-P(A\cap C)$$

$$P(A\cap C')=\frac{1}{4}-\frac{1}{8}$$

$$P(A\cap C')=\frac{2-1}{8}$$

$$P(A\cap C')=\frac{1}{8}$$

[8] If $P(A)$ = 0.6, $P(B)$ = 0.5, $P(A\cap B)$ =0.3

Compute (i) $P(A')$ (ii) $P(A\cup B)$ (iii) $P(A'\cap B)$ (iv) $P(A'\cap B')$ (v) $P(A'\cup B')$

Solution:

We have given

$P(A)$ = 0.6, $P(B)$ = 0.5, $P(A\cap B)$ =0.3

$(i)\ P(A') = 1 - P(A)$

$P(A') = 1 - 0.6$

$P(A') = 0.4$

$(ii) P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = 0.6 + 0.5 - 0.3$

$P(A \cup B) = 0.8$

$(iii) P(A' \cap B) = P(B) - P(A \cap B)$

$P(A' \cap B) = 0.5 - 0.3$

$P(A' \cap B) = 0.2$

$(iv) P(A' \cap B') = P(A \cup B)'$

$P(A' \cap B') = 1 - P(A \cup B)$

$P(A' \cap B') = 1 - 0.8$

$P(A' \cap B') = 0.2$

$(v) P(A' \cup B') = P(A \cap B)'$

$P(A' \cup B') = 1 - P(A \cap B)$

$P(A' \cup B') = 1 - 0.3$

$P(A' \cup B') = 0.7$

Multiple Choice Questions (MCQ):-

Choose the correct alternative from the following:

[1] Initially, probability was a branch of

(a) Physics (b) Statistics

(c) Mathematics (d) Economics.

Answer: (c) Mathematics

[2] Two broad divisions of probability are

(a) Subjective probability and objective probability

(b) Deductive probability and non-deductive probability

(c) Statistical probability and Mathematical probability

(d) None of these.

Answer: (a) Subjective probability and objective probability

[3] Subjective probability may be used in

(a) Mathematics (b) Statistics

(c) Management (d) Accountancy

Answer: (c) Management

[4] An experiment is known to be random if the results of the experiment

(a) Cannot be predicted

(b) Can be predicted

(c) Can be split into further experiments

(d) Can be selected at random

Answer: (d) Can be selected at random

[5] An event that can be split into further events is known as

(a) Complex event (b) Mixed event

(c) Simple event (d) Composite event

Answer: (d) Composite event

[6] Which of the following pairs of events are mutually exclusive?

(a) A: The student reads in a school B: He studies Philosophy

(b) A: Shivdeep was born in India. B: He is a fine Engineer

(c) A: Rani is 16 years old B : She is a good singer

(d) A: Peter is under 15 years of age B: Peter is a voter of Kolkata

Answer: (d) A: Peter is under 15 years of age B: Peter is a voter of Kolkata

[7] If $P(A \cap B) = 0$, then the two events A and B are

(a) Mutually exclusive (b) Exhaustive

(c) Equally likely (d) Independent.

Answer: (a) Mutually exclusive

[8] If $P(A \cup B) = 1$, then the two events A and B are

(a) Mutually exclusive (b) Equally likely

(c) Exhaustive (d) Dependent.

Answer: (c) Exhaustive

[9] If an unbiased coin is tossed once, then the two events Head and Tail are

(a) Mutually exclusive (c) Exhaustive events

(b) Equally likely (d) All these

Answer: (d) All these

[10] The probability of an event can assume any value between

(a) 1 and 1 (b) 0 and 1

(c) - 1 and 0 (d) none of these

Answer: (b) 0 and 1

[11] If P(A) = 0, then the event A

(a) will never happen (b) will always happen

(c) may happen (d) may not happen

Answer: (a) will never happen

[12] If P(A) = 1, then the event A is known as

(a) symmetric event (b) dependent event

(c) impossible event (d) sure event

Answer: (d) sure event

[13] If A, B and C are mutually exclusive and exhaustive events, then

P(A) + P(B) + P(C) equals to

(a) $\frac{1}{3}$ (b) 1

(c) 0 (d) Any value between 0 and 1

Answer: (b) 1

[14] If A denotes that a student reads in a school and B denotes that he plays cricket, then

(a) $P(A \cap B) = 1$ (b) $P(A \cup B) = 1$

(c) $P(A \cap B) = 0$ (d) $P(A) = P(B)$

Answer: (c) $P(A \cap B) = 0$

[15] Addition theorem of probability states that any two events A and B

(a) $P(A \cup B) = P(A) + P(B)$

(b) $P(A \cup B) = P(A) + P(B) + P(A \cap B)$

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(d) $P(A \cup B) = P(A) \times P(B)$

Answer: (c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

[16] For any two events A and B,

(a) $P(A) + P(B) > P(A \cap B)$

(b) $P(A) + P(B) < P(A \cap B)$

(c) $P(A) + P(B) \geq P(A \cap B)$

(d) $P(A) + P(B) \leq P(A \cap B)$

Answer: (c) $P(A) + P(B) \geq P(A \cap B)$

[17] The limitations of the classical definition of probability

(a) It is applicable when the total number of elementary events is finite

(b) It is applicable if the elementary events are equally likely

(c) It is applicable if the elementary events are mutually independent

(d) (a) and (b)

Answer: (d) (a) and (b)

[18] According to the statistical definition of probability, the probability of an event A is the

(a) Limiting value of the ratio of the no. of times the event A occurs to the number of times the experiment is repeated

(b) The ratio of the frequency of the occurrences of A to the total frequency

(c) The ratio of the frequency of the occurrences of A to the non-occurrence

of A

(d) The ratio of the favourable elementary events to A to the tolal number of elementary events.

Answer: (a)

<u>Self-Assessment Questions</u>

[1] Explain what do you understand by the term 'probability'. How the concept of probability is relevant to decision making under uncertainty?

[2] What are different approaches to the definition of probability? Are these approaches contradictory to one another? Which of these approaches you will apply for calculating the probability that:

(a) A leap year selected at random, will contain 53 Monday.

(b) An item, selected at random from a production process, is defective.

(c) Mr. Bhupinder S. Hooda will win the assembly election from Kiloi.

[3] With the help of an example explain the meaning of the following:

(i) Random experiment, and sample space

(ii) An event as a subset of sample space

(iii) Equally likely events

(iv) Mutually exclusive events.

(v) Exhaustive events

(vi) Elementary and compound events.

[4] A proof-reader is interested in finding the probability that the number of mistakes in a page will be less than 10. From his past experience he finds that out of 3600 pages he has proofed, 200 pages contained no errors, 1200 pages contained 5 errors,

and 2200 pages contained 11 or more errors. Can you help him in finding the required probability?

[5] State and develop the Addition Theorem of probability for:

(i) Mutually exclusive events

(ii) Overlapping events

(iii) Complementary events

[6] What do you understand by permutations and combinations?

(i) In how many ways we can select three players out of 12 players of the Indian Cricket team, for playing in the World XI team?

(ii) In how many ways can a sub-committee of 2 out of 6 members of the executive committee of the employees' association be constituted?

[7] What is the probability that a non-leap year, selected at random, will contain

(i) 52 Sundays? (ii) 53 Sundays? (iii) 54 Sundays?

[8] A card is drawn at random from well shuffled deck of 52 cards, find the probability that

(i) the card is either a club or diamond

(ii) the card is not a king

(iii) the card is either a face card or a club card.

[9] From a well-shuffled deck of 52 cards, two cards are drawn at random.

(a) If the cards are drawn simultaneously, find the probability that these consists of (i) both clubs, (ii) a king and a queen, (iii) a face card and a 8.

(b) If the cards are drawn one after the other with replacement. Find the probability that these consists of (i) both clubs, (ii) a king and a queen, (iii) a face card and a 8.

[10] A problem in mathematics is given to four students A, B,C, and D their chances of solving it are 1/2 , 1/3, 1/4 and 1/5 respectively. Find the probability that the problem will

(a) be solved

(b) not be solved

[11] A card is drawn at random from well shuffled deck of 52 cards, find the probability that

(a) the card is either a club or diamond

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[13] A problem in mathematics is given to four students A, B,C, and D their chances of solving it are 1/2 , 1/3, 1/4 and 1/5 respectively. Find the probability that the problem will

(a) be solved

(b) not be solved

[14] The odds that A speaks the truth are 3:2 and the odds that B does so are 7:3. In what percentage of cases are they likely to

(a) Contradict each other on an identical point?

(b) Agree each other on an identical point?

[15] Among the sales staff engaged by a company 60% are males. In terms of their professional qualifications, 70% of males and 50% of females have a degree in marketing. Find the probability that a sales person selected at random will be

(a) A female with degree in marketing

(b) a male without degree in marketing

[16] A and B play for a prize of Rs. 10,000. A is to throw a die first and is to win if he throws 1: If A fails, B it to throw and is to win if he throws 2 or 1. If B fails, A is to throw again and to win if he throws 3, 2 or 1: and so on. Find their respective expectations.

[17] A factory has three units A, B, and C. Unit A produces 50% of its products, and units B and C each produces 25% of the products. The percentage of defective items produced by A, B, and C units are 3%, 2% and 1%, respectively. If an item is selected at random from the total production of the factory is found defective, what is the probability that it is produced by:

(a) Unit A (b) Unit B (c) Unit C