MATHEMATICAL EXPECTATION (UNIVARIATE)

Introduction:

The expected value of a discrete random variable is the average value of the random variable.

If $x_1, x_2, \ldots, x_n$ are different values of a random variable x with their respective probabilities $P(x_1), P(x_2), P(x_3), \ldots\ldots, P(x_n)$ then the expected value of the random variable X is denoted by $E(X)$ and is defined as follows:

$$E(X) = x_1 \times P(x_1) + x_2 \times P(x_2) + x_3 \times P(x_3) + \ldots. + x_n \times P(x_n)$$

$$E(X) = \sum_{i=1}^{n} x_i p_i$$

Remark 1: If the p. m. f. is in functional form P(X), then

$$E(X) = \sum_{i=1}^{n} x\, P(X)$$

Remark 2: Expected value of X gives some indication as to the location of the probability distribution, although E(X) need not even be in the range of X.

For example: If X is the number of points on the uppermost face when a fair die is rolled, then

$$E(X) = \sum_{i=1}^{n} x_i p_i = \sum_{i=1}^{6} x_i P(X = x_i)$$

$$E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$$

$$E(X) = \frac{1}{6}[1 + 2 + 3 + 4 + 5 + 6]$$

$$E(X) = \frac{21}{6} = 3.5$$

Probability as an Expectation:

Let A be any event. We can write P(A) as an expectation, as follows.

Define the indicator function:

$$I_A = \begin{pmatrix} 1 & \text{if event A occurs} \\ 0 & \text{otherwise} \end{pmatrix}$$

Then I_A is a random variable, and

$$E(I_A) = \sum_{r=0}^{1}(X = r)\, P(I_A = r) = 0 \times P(I_A = 0) + 1 \times P(I_A = 1)$$

$$E(I_A) = P(I_A = 1) = P(A).$$

Thus $P(A) = E(I_A)$ for any event A

Expected Value of a Function of a Random Variable:-

Let X be a discrete random variable with p. m. f. P(X), and let Y=g(X). Suppose that we are interested in finding E(Y). One way to find E(Y) is to first find the p. m. f. of Y and then use the expectation of Y as follows:

$$E(Y) = E\left[g(X)\right] = \sum_{i=1}^{n} g(X)P(X)$$

Theorems on Expectation:

Theorem 1: Expected value of a constant is the constant itself. E(C) = C

Proof: Let (x_i, p_i); i = 1, 2, 3, ..., n denote the probability distribution of a discrete random variable X.

Let g(X) = C, is a constant

$$E(C) = E\left[g(X)\right] = \sum_{i=1}^{n} g(x_i)p_i$$

$$E(C) = \sum_{i=1}^{n} C \times p_i$$

$$E(C) = C\sum_{i=1}^{n} p_i \qquad (\because C \text{ is constant})$$

$$E(C) = C\ (1) \qquad \left(\because \sum_{i=1}^{n} p_i = 1\right)$$

$$E(C) = C$$

Theorem 2: The mathematical expectation of the sum of function of a random variable and the other constant is equal to the sum of the mathematical expectation of the function of that random variable and the other constant i.e.

E(X + b) = E(X) + b

Proof: Let (x_i, p_i); i = 1, 2, 3, ..., n denote the probability distribution of a discrete random variable X.

Let g(X) = X + b

$$E(X+b)=E[g(X)]=\sum_{i=1}^{n} g(x_i)p_i$$

$$E(X+b)=\sum_{i=1}^{n}(x_i+b) \times p_i$$

$$E(X+b)=\sum_{i=1}^{n} x_i\ p_i + b\sum_{i=1}^{n} p_i \quad (\because b \text{ is constant})$$

$$E(X+b)=E(X)+b(1) \quad \left(\because \sum_{i=1}^{n} x_i\ p_i = E(X) \text{ and } \sum_{i=1}^{n} p_i = 1\right)$$

$$E(X+b)=E(X)+b$$

Theorem 3: The mathematical expectation of the sum of product between a constant and function of a random variable is equal to the sum of the product of the constant and the mathematical expectation of the function of that random variable i.e. E(a X) = a E(X)

Proof: Let (x_i, p_i); i = 1, 2, 3, ..., n denote the probability distribution of a discrete random variable X.

Let g(X) = a X

$$E(a\,X)=E[g(X)]=\sum_{i=1}^{n} g(x_i)p_i$$

$$E(a\,X)=\sum_{i=1}^{n}(ax_i) \times p_i$$

$$E(a\,X)=a\sum_{i=1}^{n} x_i\ p_i \qquad (\because a \text{ is constant})$$

$$E(a\,X)= aE(X) \qquad \left(\because \sum_{i=1}^{n} x_i\ p_i = E(X)\right)$$

$$E(a\,X)= aE(X)$$

Theorem 4: The mathematical expectation of the sum of product between a constant and function of a random variable and the other constant is equal to the sum of the product of the constant and the mathematical expectation of the function of that random variable and the other constant.

i.e. E(a X +b) = a E(X) + b

Proof: Let (x_i, p_i); i = 1, 2, 3, ..., n denote the probability distribution of a discrete random variable X.

Let g(X) = a X + b

$$E(a\,X+b)=E[g(X)]=\sum_{i=1}^{n} g(x_i)p_i$$

$$E(a\,X+b)=\sum_{i=1}^{n}(ax_i+b)\ \times\ p_i$$

$$E(a\,X+b)=\sum_{i=1}^{n}ax_i\times\ p_i+\sum_{i=1}^{n}b\ p_i$$

$$E(a\,X+b)=a\sum_{i=1}^{n} x_i\ p_i+b\sum_{i=1}^{n} p_i \qquad (\because a \text{ is constant})$$

$$E(a\,X+b)= aE(X)+b(1) \qquad \left(\because \sum_{i=1}^{n} x_i\ p_i = E(X) \text{ and } \sum_{i=1}^{n} p_i = 1\right)$$

$$E(a\,X+b)= aE(X)+b$$

Variance and Standard Deviation of a Random Variable:

The variance of a random variable X is a measure of how spread out it is. Are the values of X clustered tightly around their mean, or can we commonly observe values of X a long way from the mean value? The variance measures how far the values of X are from their mean, on average.

Definition: Variance of a random variable X is defined as the Arithmetic Mean of the Square of Deviations taken about Arithmetic Mean

$$\text{Var}(X) = \sigma^2 = E[X-E(X)]^2 = E[X-\mu]^2$$

The variance is the mean squared deviation of a random variable from its own mean.

If X has high variance, we can observe values of X a long way from the mean.

If X has low variance, the values of X tend to be clustered tightly around the mean value.

Theorems on Variance:

Theorem 1: Variance of a constant zero. V(C) = 0; C is constant

Proof: Let (x_i, p_i); i = 1, 2, 3, ..., n denote the probability distribution of a discrete random variable X.

By definition of variance

$$\text{Var}(X) = E[X-E(X)]^2$$

$$\text{Var}(C) = E[C-C]^2 \qquad \begin{pmatrix} \because C \text{ is constant} \\ E(C) = C \end{pmatrix}$$

$$\text{Var}(C) = E[0]^2$$

$$\text{Var}(C) = 0$$

Theorem 2: Variance is invariant to the change of origin. i. e. V(X + a) = V(X);

a is constant.

Proof: Let (x_i, p_i); i = 1, 2, 3, ..., n denote the probability distribution of a discrete random variable X.

By definition of variance

$$\mathrm{Var}(X+a) = E\left[(X+a)-E(X+a)\right]^2$$

$$\mathrm{Var}(X+a) = E\left[X+a-E(X)-E(a)\right]^2$$

$$\mathrm{Var}(X+a) = E\left[X+a-E(X)-a\right]^2 \begin{pmatrix} \because \text{ a is constant} \\ E(a)=C \end{pmatrix}$$

$$\mathrm{Var}(X+a) = E\left[X-E(X)\right]^2$$

$$\mathrm{Var}(X+a) = V(X) \qquad \begin{pmatrix} \because \text{ by definition of variance} \\ V(X)=E\left[X-E(X)\right]^2 \end{pmatrix}$$

Theorem 3: Variance is invariant to the change of origin. i. e. $V(aX) = a^2V(X)$; a is constant.

Proof: Let (x_i, p_i); i = 1, 2, 3, ..., n denote the probability distribution of a discrete random variable X.

By definition of variance

$$\mathrm{Var}(aX)= E\left[(aX)-E(aX)\right]^2$$

$$\mathrm{Var}(aX) = E\left[aX-aE(X)\right]^2 \qquad (\because E(a)=a)$$

$$\mathrm{Var}(aX) = E\left\{a\left[X-E(X)\right]^2\right\}$$

$$\mathrm{Var}(aX) = a^2\left\{E\left[X-E(X)\right]^2\right\}$$

$$\mathrm{Var}(aX) = a^2V(X) \qquad \begin{pmatrix} \because \text{ by definition of variance} \\ V(X)=E\left[X-E(X)\right]^2 \end{pmatrix}$$

Theorem 4: Variance is invariant to the change of origin. i. e. $V(aX+b) = a^2V(X)$; a and b are constants.

Proof: Let (x_i, p_i); i = 1, 2, 3, ..., n denote the probability distribution of a discrete random variable X.

By definition of variance

$$\mathrm{Var}(aX+b) = E\left[(aX+b)-E(aX+b)\right]^2$$

$$\mathrm{Var}(aX+b) = E\left[aX+b-E(aX)-E(b)\right]^2$$

$$\mathrm{Var}(aX+b) = E\left[aX+b-aE(X)-b\right]^2 \begin{pmatrix} \because E(b)=b \\ E(aX)=aE(X) \end{pmatrix}$$

$$\mathrm{Var}(aX+b) = E\left[aX-aE(X)\right]^2$$

$$\mathrm{Var}(aX+b) = E\left\{a^2\left[X-E(X)\right]^2\right\}$$

$$\mathrm{Var}(aX+b) = a^2\left\{E\left[X-E(X)\right]^2\right\}$$

$$\mathrm{Var}(aX+b) = a^2 V(X) \qquad \begin{pmatrix} \because \text{by definition of variance} \\ V(X)=E\left[X-E(X)\right]^2 \end{pmatrix}$$

Remark:

(i) If we define $y = \frac{x-a}{h}$ then $\sigma_Y^2 = \frac{1}{h^2}\sigma_x^2$

$$h^2\sigma_Y^2 = \sigma_x^2$$

$$\sigma_x^2 = h^2\sigma_Y^2$$

$$\sigma_x = |h|\sigma_y$$

(ii) If we define $Y = \frac{x-\mu}{\sigma}$, μ & σ are mean & standard deviation respectively.

$$\therefore E(Y) = E(x)-E(\mu)$$

$$E(Y) = \frac{\mu-\mu}{\sigma} = 0$$

$$\mathrm{Var}(y) = \mathrm{Var}\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma^2}\mathrm{Var}(X) = \frac{1}{\sigma^2}\sigma^2 = 1$$

Y has mean 0 and variance 1 is called standardized r. v.

(iii) $\mathrm{Var}\,(-3X-5) = (-3)^2\,\mathrm{Var}(X) = 9\,\mathrm{Var}(x)$

(iv) S.D. $\left(-3X-5\right)=\left|-3\right|S.D.\left(X\right)$ = $3S.D.\left(X\right)$

Examples:

[1] The mean and variance of marks in statistics (x) are 60 and 25 respectively. Find the mean and variance of $Y=\frac{X-60}{5}$

Solution: – E(x) = 60 & Var(X) = 25

$$Y=\frac{X-60}{5}$$
$$E(Y)=\frac{E(X)-60}{5}=0$$

$$Var\left(Y\right)=\frac{1}{25}Var\left(X\right)$$
$$Var\left(Y\right)=\frac{1}{25}25$$
$$Var\left(Y\right)=1$$

[2] Let X be a discrete r.v. with mean 5 and S. d. 3. Compute mean and S. d. of 2X-5, 3-7x, $\frac{X+1}{2}$

Solution: – Given E(x) =5, $\sigma_x=3$

Let Y= 2X-5

E(Y) = 2 E(X) -5 = 2 x 5 -5= 5

S.D. $\left(2X-5\right)=\left|2\right|S.D.\left(X\right)$ = 2×3 = 6

Let Y= 3-7X

E(Y) = 3-7 E(X) = 3-7 x 5= -32

S.D. $\left(3-7X\right)=\left|-7\right|S.D.\left(X\right)$ = 7×3 = 21

Let $Y = \frac{X+1}{2}$

$$E(Y) = \frac{E(X)+1}{2}$$
$$E(Y) = \frac{5+1}{2}$$
$$E(Y) = 3$$

$$S.D.\left(Y = \frac{X+1}{2}\right) = \left|\frac{1}{2}\right| S.D.(X) = \frac{1}{2}\times 3 = \frac{3}{2}$$

Moments of a random variable:-

Raw moment (Definition):-

The r^{th} raw moment of X is defined as the r^{th} moment about zero. It is denoted by μ_r' and is given by

$$\mu_r' = E(x^r) = \sum_{i=1}^{n} x_i^r . p_i, \qquad r = 1,2,3,....n$$
$$\mu_0' = 1 \quad \text{and} \quad \mu_1' = \text{Mean}$$

Central moment (Definition):-

The r^{th} central moment of X is defined as the r^{th} moment about mean. It is denoted by μ_r and is given by

$$\mu_r = E[X-E(X)]^r = \sum_{i=1}^{n} E[X_i - E(X)]^r . P_i, \qquad r = 1,2,3,....n$$
$$\mu_0 = 1 \qquad \mu_1 = 0$$
$$\mu_2 = E[X-E(X)]^2 = var(X)$$

Moment about any arbitrary point ‘a’(Definition):-

The r^{th} arbitrary moment of X is defined as the r^{th} moment about 'a'. It is denoted by $\mu_r^{'}(a)$ and is given by

$$\mu_r^{'}(a)=E(X-a)^r=\sum_{i=1}^{n}(X_i-a)^r.P_i$$

Note: Taylor series for

$e^x \ \forall x \in R$

[i] $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{k=0}^{\infty}\frac{x^k}{k!}$

[ii] $(x+y)^2 = x^2 + y^2 + 2xy$

[iii] $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2zx + 2yz$

Where, $M_x(t)$ is finite and $t \in [-a, a]$, a is positive constant

Moment generating function (M.G.F.):-

Moment is useful to find moments of probability distribution. It is also useful in distribution theory. If two random variables have same m.g.f. then they have the same distribution.

Definition:-

Suppose X is a random variable with p.m.f. P(X) then the m.g.f. of X is denoted by $M_X(t)$ and is given by

$$M_X(t)=E(e^{tx})=\sum e^{tx}.P(x)$$

Provided $E(e^{tx})$ is convergent for the values of t in neighborhood of zero $(i.e. -h < t < h,\ h > 0)$. $M_X(t)$ can be expressed in powers of t as follows

$$M_X(t)=E(e^{tx})=E(1+tx+\frac{(tx)^2}{2!}+\frac{(tx)^3}{3!}+\frac{(tx)^4}{4!}+......)$$

$$M_X(t)=E(1+tx+\frac{t^2x^2}{2!}+\frac{t^3x^3}{3!}+\frac{t^4x^4}{4!}+....................)$$

$$M_X(t)=1+t.E(x)+\frac{t^2}{2!}.E(X^2)+\frac{t^3}{3!}E(x^3)+\frac{t^4}{4!}E(x^4)+...........................$$

$$M_X(t)=1+t\mu_1'+\frac{t^2}{2!}.\mu_2'+\frac{t^3}{3!}.\mu_3'+\frac{t^4}{4!}\mu_4'+...........$$

$$M_X(t)=\sum_{r=0}^{\infty}\mu_r'\frac{t^r}{r!}$$ (we conclude that the r^{th} moment of x is the coefficient of $\frac{t^r}{r!}$

Properties of Moment Generating Function (m. g. f.):

[1] $M_x(0)=1$

Proof: By definition of m. g. f.

$$M_X(t)=E(e^{tx})$$
$$\therefore M_X(0)=E(e^{0x})=E(e^0)=E(1)=1$$

[2] If X is r. v. with m. g. f. of $M_X(t)$ and a is constant then prove that $M_{x+a}(t)=e^{at}M_X(t)$

Proof: By definition of m. g. f.

$$M_{X+a}(t)=E\left[e^{t(x+a)}\right]=E\left[e^{tx+at}\right]=E\left[e^{tx}.e^{at}\right]$$
$$=e^{at}.E\left[e^{tx}\right]=e^{at}M_X(t)$$

[3] If $M_X(t)$ is a m. g. f. of a r. v. X then prove that: $M_{cx}(t)=M_x(ct)$, c is constant

Proof: By definition of m.g.f.

$$M_X(t)=E(e^{tx})$$
$$\therefore M_{CX}(t)=E\left[e^{tCX}\right]=E\left[e^{(Ct)X}\right]=M_X(ct)$$

[4] If $M_X(t)$ is a m. g. f. of a r. v. X then prove that $M_{a+cX}(t)=e^{at}M_X(ct)$

Proof: By definition of m.g.f.

$$M_X(t)=E(e^{tx})$$
$$\therefore M_{a+cx}(t)=E[e^{t(a+cx)}]=E[e^{at+cxt}]$$
$$M_{a+cx}(t)=E[e^{at}.e^{cxt}]=e^{at}.E[e^{ctx}]=e^{at}M_X(ct)$$

[5] If $M_X(t)$ is a m. g. f. of a r. v. X then prove that: $M_{\left(\frac{X-a}{b}\right)}(t)=e^{\frac{-at}{b}}.M_X\left(\frac{t}{b}\right)$

[6] If X & Y are independent random variable with M.G.F. $M_X(t)$ and $M_Y(t)$ respectively then $M_{X+Y}(t)=M_X(t).M_Y(t)$

Proof:

$$M_{x+y}(t)=E[e^{xt+ty}]=E[e^{tx}.e^{ty}]=E[e^{tx}].E(e^{ty}) \qquad (\because \text{X \& Y are independent r.v.})$$
$$M_{x+y}(t)=M_X(t).M_Y(t)$$

Uniqueness property of M.G.F:

For a given probability distribution there is unique M.G.F. if it exists and for a given M.G.F. there is a unique probability distribution

Cumulative generating function (C.G.F) :

It is useful to find the central moments

Definition: – In a given random variable X has M.G.F. $M_X(t)$ then the cumulative generating function is denoted by $K_X(t)$ and is given by

$$K_X(t)=\log_e M_X(t)$$

$$K_X(t)=k_1t+k_2\frac{t^2}{2!}+k_3\frac{t^3}{3!}+....+k_r\frac{t^r}{r!}$$

$$K_X(t)=\log_e\left(1+\mu_1^{'}t+\mu_2^{'}\frac{t^2}{2!}+....+\mu_r^{'}\frac{t^r}{r!}\right)$$

$$=\left\{\begin{array}{l}\left[\mu_1^{'}t+\mu_2^{'}\frac{t^2}{2!}+\mu_3^{'}\frac{t^3}{3!}+.........\right]-\frac{1}{2}\left[\mu_1^{'}t+\mu_2^{'}\frac{t^2}{2!}+\mu_3^{'}\frac{t^3}{3!}+..........\right]^2\\+\frac{1}{3}\left[\mu_1^{'}t+\mu_2^{'}\frac{t^2}{2!}+\mu_3^{'}\frac{t^3}{3!}+.........\right]^3-\frac{1}{4}\left[\mu_1^{'}t+\mu_2^{'}\frac{t^2}{2!}+\mu_3^{'}\frac{t^3}{3!}+.........\right]^4+..........-\end{array}\right\}$$

Comparing a coefficients of like powers of t on both sides, we get, the relationship between the moments and cumulants.

$$k_1=\mu_1^1=\text{mean}$$

$$\frac{k_2}{2!}=\frac{\mu_2^1}{2!}-\frac{(\mu_1^1)^2}{2}=k_2\mu_2^1-(\mu_1^1)^2=\mu_2$$

$$\frac{k_3}{3!}=\frac{\mu_3^1}{3!}-\frac{2}{2}\frac{\mu_1^1\mu_2^1}{2!}+\frac{1}{3!}\left(\mu_1^1\right)^3$$

$$k_3=\mu_3^1-3\mu_1^1\mu_2^1+2\left(\mu_1^1\right)^3=\mu_3$$

$$k_4=\mu_4-3k_2{}^2$$

$$\mu_4=k_4+3k_2{}^2$$

If X and Y two r.v. and equality holds for their m.g.f. $M_x(t)=M_y(t)$ then X and Y have the same probability distribution $F_X(x)=F_Y(y)$

Note:

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-...........|x|<1$$

$$(x+y+z)^3=x^3+y^3+2^3+3(x+y)(y+2)(x+2)$$

Properties of cumulants generating function (c. g. f.):

[1] If Y=X-a, then except the first cumulant all other cumulants of X and Y are same, a being constant.

Proof: Let $M_X(t)$ be M.G.F. of X, therefore, we have property M.G.F.

$$M_{X+a}(t) = e^{at}M_X(t)$$
$$M_{x-a}(t) = e^{-at}M_x(t)$$

Given Y = X-a

$$M_Y(t) = M_{X-a}(t) = e^{-at}M_X(t)$$

Taking log on the both side, we get

$$\log M_Y(t) = \log[e^{-at}M_X(t)]$$
$$K_Y(t) = -at + \log M_X(t)$$
$$K_Y(t) = -at + K_X(t)$$

$$K_1(t)+K_2\frac{t^2}{2!}+K_3\frac{t^3}{3!}+........=-at\left[K_1(t)+K_2\frac{t^2}{2!}+K_3\frac{t^3}{3!}+........\right]$$

$$K_1(t)+K_2\frac{t^2}{2!}+K_3\frac{t^3}{3!}+........= t\left(K_1-a\right)+K_2\frac{t^2}{2!}+K_3\frac{t^3}{3!}+........$$

Equating coefficient of t on both side, we get

First cumulant is $k_1 = k_1 - a$ and all other cumulants $r \geq 2$ are same.

[2] If Y = hX then , r cumulants of $Y = h^r \times (r^{th}\text{cumulant of X})$, h is constant.

Solution: Let $M_X(t)$ be m.g.f. of X, given Y= h X

$\therefore M_y(t) = M_X(ht)$ $\left(\because \text{by property of m.g.f.}\right)$

Taking log on both sides, we get

$$\log M_Y(t) = \log M_X(ht)$$

$$\therefore K_Y(t) = K_X(t) \ldots\ldots\ldots\ldots\ldots\ldots (\because \text{by defination of cumulants})$$

$$K^1 + K^2 \frac{t^2}{2!} + K^3 \frac{t^3}{3!} + \ldots\ldots = K_1(ht) + K_2 \frac{(ht)^2}{2!} + \ldots\ldots\ldots + K_r \frac{(ht)^r}{r!}$$

$$K_1 t + K_2 \frac{t^2}{2!} + K_3 \frac{t^3}{3!} + \ldots\ldots\ldots + \frac{t^r}{r!} K_r = h.K_1 + h^2 K_2 \frac{t^2}{2!} + h^3 K_3 \frac{t^3}{3!} + \ldots\ldots + h^r K_r \frac{t^r}{r!}$$

Equating coefficients of $\frac{t^r}{r!}$ on both sides we get

$$K_r \text{ of } Y = h^r.K_r \text{ of } X$$

(3) Additive property of cumulants:

If X and Y are independent random variables then K_r of X+K_r of Y

Proof:

Let $M_X(t)$ and $M_Y(t)$ be m. g. f of X and Y

Given X and Y are independent random variable then $M_{X+Y}(t) = M_X(t) M_Y(t)$

Taking log on both sides we get

$$\log\left(M_{X+Y}(t)\right) = \log\left[M_X(t) M_Y(t)\right]$$

$$\log M_{X+Y}(t) = \log M_X(t) + \log M_Y(t)$$

$$K_{X+Y}(t) = K_X(t) + K_Y(t)$$

Equating coefficients of $\frac{t^r}{r!}$ on both sides, We get

K_r of (X+Y)=K_r of X +K_r of Y

Coefficient of skewness (Y_1) is defined as

$$\gamma_1 = \sqrt{\beta} = \sqrt{\frac{\mu_3^{\,2}}{\mu_2^{\,3}}} = \frac{\mu_3}{\sqrt{(\sigma^2)^3}} = \frac{\mu_3}{\sigma^3}$$

Interpretation:

If $\gamma_1 = 0$, the distribution is symmetric

If $\gamma_1 > 0$, the distribution is positively skewed

If $\gamma_1 < 0$, the distribution is negatively skewed

The coefficient of kurtosis (γ_2) is defined as

$$\gamma_2 = \beta_2 - 3 = \frac{\mu_4}{\mu_2^{\,2}} - 3$$

It is also called as excess of kurtosis

Interpretation:

If $\gamma_2 = 0$, the distribution is mesokurtic

If $\gamma_2 > 0$, is the distribution is leptokurtic

If $\gamma_2 < 0$, is the distribution is platykurtic

Theory Questions:

[1] Define mathematical expectation of a discrete random variable X

[2] Explain how E (X) is the arithmetic mean of X. Can E (X) always be one of the possible values of X? Explain.

[3] What is the physical interpretation of E (X)?

[4] Define expectation of a function of random variable.

[5] Define variance of a discrete random variable.

[6] Show that variance is invariant to the change of origin, but not of scale.

[7] What is meant by standardized random variable? Explain with the help of an illustration.

[8] Prove that Variance of a constant is zero,

[9] Define moment generating function of a random variable X. prove that $M_X(0)=1$

[10] Define moment generating function of a random variable X. prove that $M_{X+a}(t)= e^{at}M_X(t)$

[11] Explain how the raw moments can be obtained using M.G.F.

[12] Define cumulant generating function of r.v. X. Also explan how to obtain the cumulants from C.G.F.

[13] State expressions for four central moments in terms of first four cumulants.

[14] State the properties of C.G.F.

[15] Show that the C.G.F. of sum of two independent r.v.s. is sum of their C.G.F's.

[16] Prove that the central moments are invariant to the change of origin.