CONDITIONAL PROBABILITY AND BAYES THEOREM

Introduction:

Independent Event of Three Events:

Two events are said to be **independent** of each other if the occurrence or non-occurrence of one event in any trial does not affect the occurrence of the other event in any trial.

Two events A and B are said to be **independent** of each other if and only if the following three conditions hold:

Conditions for the independence of two events A and B:

$P(A/B) = P(A) \quad (i)$

$P(B/A) = P(B) \quad (ii)$

$P(A \cap B) = P(A) \times P(B) \quad (iii)$

The first two equations have a clear, intuitive appeal. The top equation says that when A and B are independent of each other, then the probability of A stays the same even when we know that B has occurred - it is a simple way of saying that knowledge of B tells us nothing about A when the two events are independent. Similarly, when A and B are independent, then knowledge that A has occurred gives us absolutely no information about B and its likelihood of occurring.

The third equation, however, is the most useful in applications. It tells us that when A and B are independent (and only when they are independent), we can obtain the probability of the joint occurrence of A and B (i.e. the probability of their intersection) simply by multiplying the two separate probabilities. This rule is thus called the **Product Rule for Independent**

Theorem-1: Suppose A and B are two events defined on a sample space S. If A and B are independent then prove that A and B' are independent.

Proof: Given that A and B are independent then

$$P(A \cap B) = P(A) \times P(B) \qquad (i)$$

We have to prove that A and B’ are independent.

Now,

$$P(A \cap B') = P(A) - P(A \cap B)$$

$$P(A \cap B') = P(A) - P(A)P(B) \qquad [\because (i)]$$

$$P(A \cap B') = P(A)[1 - P(B)]$$

$$P(A \cap B') = P(A) \times P(B')$$

$\therefore$ A and B’ are independent

Theorem-2: Suppose A and B are two events defined on a sample space S. If A and B are independent then prove that A’ and B are independent.

Proof: Given that A and B are independent then

$$P(A \cap B) = P(A) \times P(B) \qquad (i)$$

We have to prove that A’ and B are independent.

Now,

$$P(A' \cap B) = P(B) - P(A \cap B)$$

$$P(A' \cap B) = P(B) - P(A)P(B) \qquad [\because (i)]$$

$$P(A' \cap B) = P(B)[1 - P(A)]$$

$$P(A' \cap B) = P(A') \times P(B)$$

$\therefore$ A’ and B are independent

Theorem-3: Suppose A and B are two events defined on a sample space S. If A and B are independent then prove that A’ and B’ are independent.

Proof: Given that A and B are independent then

$$P(A \cap B) = P(A) \times P(B) \qquad (i)$$

We have to prove that A’ and B’ are independent.

Now,

$$P(A' \cap B') = P(A \cup B)'$$

$$P(A' \cap B') = 1 - P(A \cup B)$$

$P(A' \cap B') = 1-[P(A)+P(B)-P(A \cap B)]$

$P(A' \cap B') = 1-P(A)-P(B)+P(A \cap B)$

$P(A' \cap B') = 1-P(A)-P(B)+P(A) \times P(B) \qquad [\because (i)]$

$P(A' \cap B') = [1-P(A)]-P(B)[1-P(A)]$

$P(A' \cap B') = [1-P(A)][1-P(B)]$

$P(A' \cap B') = P(A') \times P(B')$

$\therefore$ A' and B' are independent

Conditions for the independence of three events A, B and C:

Mutual Independence:

Definition:- Three events A, B and C are said to be mutually independence or completely independent if and only if the following conditions are satisfied:

$P(A \cap B) = P(A) \times P(B) \qquad (i)$

$P(B \cap C) = P(B) \times P(C) \qquad (ii)$

$P(A \cap C) = P(A) \times P(C) \qquad (iii)$

$P(A \cap B \cap C) = P(A) \times P(B) \times P(C) \qquad (iv)$

Pairwise Independence:

Definition:- Three events A, B and C are said to be pairwise independence if and only if the following conditions are satisfied:

$P(A \cap B) = P(A) \times P(B) \qquad (i)$

$P(B \cap C) = P(B) \times P(C) \qquad (ii)$

$P(A \cap C) = P(A) \times P(C) \qquad (iii)$

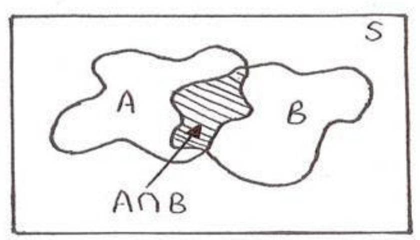
Remark:

[i] A, B and C are mutually independent then A, B and C are pairwise independent.

The Conditional Probability Rule:

As a measure of uncertainty, probability depends on information. We often face situations where the probability of an event A is influenced by the information that another event B has occurred. Thus, the probability we would give the event "Xerox stock price will go up tomorrow" depends on what we know about the company and its performance; the probability is conditional upon our information set. If we know much about the company, we may assign a different probability to the event than if we know little about the company. We may define the probability of event A conditional upon the occurrence of event B. In this example, event A may be the event that the stock will go up tomorrow, and event B may be a favourable quarterly report.

Consider two events A and B defined over the sample space *S*, as shown in Figure



Conditional Probability:-

[1] Conditional probability of A given B

Let A and B be two events defined on a sample space Ω of a random experiment then the conditional probability of A given B is denoted by $P(A/B)$

$$P(A/B)=\frac{P(A\cap B)}{P(B)}, \qquad P(B) > 0$$

[2] Conditional probability of B given A

Let A and B be two events defined on a sample space Ω of a random experiment then the conditional probability of B given A is denoted by $P(B/A)$

$$P(B/A)=\frac{P(A\cap B)}{P(A)}, \qquad P(A)>0$$

The vertical line in $P(A/B)$ is read given, or conditional upon. Therefore, the probability of event A given the occurrence of event B is defined as the probability of the intersection of A and B, divided by the probability of event B.

Remarks:

[i] If A and B are mutually exclusive events defined on Ω then

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{P(\phi)}{P(B)}=0$$

[ii] $P(A/A')=\frac{P(A\cap A')}{P(A')}=\frac{P(\phi)}{P(A')}=0$

[iii] If $B\subset A$ then $P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{P(B)}{P(B)}=1$

The Product Rule:

The **Product Rule** (also called **Multiplication Theorem)** allows us to write the probability of the simultaneous occurrence of two (or more) events.

In the conditional probability rules

$$P(A/B)=\frac{P(A\cap B)}{P(B)}, \qquad P(B)>0$$

$$P(B/A)=\frac{P(B\cap A)}{P(A)}, \qquad P(A)>0$$

$(A\cap B)$ or $(B\cap A)$ is the event A and B occur simultaneously. So, rearranging the conditional probability rules, we have our product rule.

$P(A\cap B)=P(A/B)\times P(B)$ and $P(B\cap A)=P(B/A)\times P(A)$.

The Product Rule states that the probability that both A and B will occur simultaneously is equal to the probability that B (or A) will occur multiplied by the conditional probability that A (or B) will occur, when it is known that B (or A) is certain to occur or has already occurred.

<u>Baye’s Theorem:</u>

As we have already noted in the introduction, the basic objective behind calculating probabilities is to help us in making decisions by quantifying the uncertainties involved in the situations. Quite often, whether it is in our personal life or our work life, decision-making is an ongoing process. Consider for example, a seller of winter garments, who is interested in the demand of the product. In deciding on the amount he should stock for this winter, he has computed the probability of selling different quantities and has noted that the chance of selling a large quantity is very high. Accordingly, he has taken the decision to stock a large quantity of the product. Suppose, when finally the winter comes and the season ends, he discovers that he is left with a large quantity of stock. Assuming that he is in this business, he feels that the earlier probability calculation should be updated given the new experience to help him decide on the stock for the next winter.

Similar to the situation of the seller of winter garment, situations exist where we are interested in an event on an ongoing basis. Every time some new information is available, we do revise our odds mentally. This revision of probability with added information is formalised in probability theory with the help of famous **Bayes' Theorem**. The theorem, discovered in 1761 by the English clergyman Thomas Bayes, has had a profound impact on the development of statistics and is responsible for the emergence of a new philosophy of science. Bayes himself is said to have been unsure of his extraordinary result, which was presented to the Royal Society by a friend in 1763 - after Bayes' death. We will first understand **The Law of Total**

Probability, which is helpful for derivation of Bayes' Theorem.

The Law of Total Probability:-

Consider two events A and B. Whatever may be the relation between the two events, we can always say that the probability of A is equal to the probability of the intersection of A and B, plus the probability of the intersection of A and the complement of B (event B).

Statement:

If $E_1, E_2 \ldots, E_i, \ldots.., E_n$ are mutually exclusive events defined on Ω with $P(E_i) \neq 0$ (i=1,2,3,...) then for any arbitrary events A which is a proper subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0,$ then prove that

$$P(E_i/A) = \frac{P(E_i).\ P(A/E_i)}{P(A)} = \frac{P(E_i).\ P(A/E_i)}{\sum_{i=1}^{n} P(E_i).\ P(A/E_i)}, \text{ (i=1,2,3....n)}$$

Proof: Bayes' theorem is easily derived from the law of total probability and the definition of conditional probability.

From Venn diagram $(A \cap E_1), (A \cap E_2), \ldots.., (A \cap E_i), \ldots.., (A \cap E_n)$ are mutually exclusive events. By the law of total probability, we have

$$A = (A \cap E_1)U(A \cap E_2)U,.....,U(A \cap E_i),.....,U(A \cap E_n)$$

$$P(A) = P[(A \cap E_1)U(A \cap E_2)U,.....,U(A \cap E_i),.....,U(A \cap E_i)]$$

$$P(A) = P[(A \cap E_1)+(A \cap E_2)+,.....,+(A \cap E_i),.....,+(A \cap E_i)] \quad ......\left[\because \text{axiom}(\text{iii})\right]$$

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i)...........(1)$$

By defination of conditional probability A given E_i

$$P(A/E_i) = \frac{P(A \cap E_i)}{P(E_i)}, \ P(E_i) > 0$$

$$\Rightarrow P(A \cap E_i) = P(A/E_i) \times P(E_i).....(2)$$

Similarly, conditional probability of E_i given A

$$P(E_i/A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i \cap A)}{P(A)}, \ P(A) > 0$$

$$\Rightarrow P(A \cap E_i) = P(E_i/A) \times P(A)...........(3)$$

From (2) & (3), we get

$$P(A/E_i) \times P(E_i) = P(E_i/A) \times P(A)$$

$$\Rightarrow P(E_i/A) = \frac{P(A/E_i) \times P(E_i)}{P(A)}$$

But from (1), we get

$$P(E_i/A) = \frac{P(A/E_i) \times P(E_i)}{\sum_{i=1}^{n} P(A \cap E_i)}$$

$$P(E_i/A) = \frac{P(A/E_i) \times P(E_i)}{\sum_{i=1}^{n} P(A/E_i) \times P(A)}.......(\because (2))$$

Hence, proved

The theorem gives the probability of one of the sets in the partition $E_{i,}$ given the occurrence of event A.

Thus the theorem allows us to reverse the conditionality of events: we can obtain the probability of E given A from the probability of A given E (and other information).

As we see from the theorem, the probability of E given A is obtained from the probabilities of E and $\bar{E}$ rom the conditional probabilities of A given E and A given $\bar{E}$.

[i] The probabilities P(E) and P($\bar{E}$) are called **prior probabilities** of the events E and $\bar{E}$;

[ii] The probability P(E /A) is called the **posterior probability** of E. It is possible to write Bayes' theorem in terms of $\bar{E}$ and A, thus giving the posterior probability of $\bar{E}$, P($\bar{E}$ /A).

[iii] Bayes' theorem may be viewed as a means of transforming our prior probability of an event E into a posterior probability of the event E - posterior to the known occurrence of event A.

[iv] The Bayes' theorem can be extended to a partition of more than two sets.

Application of Bayes Theorem

[1] If A and B are independent event with $P(A)$ = 0.6, $P(B)$ = 0.3

Find $(i)P(A \cup B)$ $(ii)P(A \cap B')(iii)P(A' \cap B)(iv)P(A' \cap B')$

Solution: If A and B are independent event then

$$P(A \cap B) = P(A) \times P(B)$$
$$P(A \cap B) = 0.6 \times 0.3$$
$$P(A \cap B) = 0.18$$

$$(i)P(A \cup B) = P(A)+P(B)-P(A \cap B)$$
$$P(A \cup B) = 0.6 + 0.3 - 0.18$$
$$P(A \cup B) = 0.72$$

$(ii) P(A \cap B') = P(A) \times P(B')$

$P(A \cap B') = 0.6 \times 0.7$

$P(A \cap B') = 0.42$

[2] Suppose a card is drawn at random from a well-shuffled pack of 52 playing cards. Let event A is a getting a spade card and B is a getting a king. Are A and B independent?

Solution:

$$P(A) = \frac{13}{52} = \frac{1}{4} \quad , \quad P(B) = \frac{4}{52} = \frac{1}{13} \quad \text{and} \quad P(A \cap B) = \frac{1}{52} \cdots (i)$$

Now,

$$P(A) \times P(B) = \frac{1}{4} \times \frac{1}{13}$$

$$P(A) \times P(B) = \frac{1}{52} \cdots (ii)$$

From (i) and (ii)

$$P(A \cap B) = P(A) \times P(B)$$

And therefore A and B are independent

[3] Let A and B are two events connected with a random experiement such that

$P(A) = \frac{1}{2}, P(A \cup B) = \frac{3}{4}$ and $P(B') = \frac{5}{8}$. Find

$(i) P(A \cap B) \ (ii) P(B \cap A') (iii) P(A' \cup B') (iv) P(A' \cap B') (v) P(B' \cap A)$

Solution: We have $P(A) = \frac{1}{2}, P(A \cup B) = \frac{3}{4}$ and $P(B') = \frac{5}{8}$

$$P(B) = 1 - P(B') = 1 - \frac{5}{8} = \frac{3}{8}$$

$$(i) P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$
$$P(A \cap B) = \frac{1}{2} + \frac{3}{8} - \frac{3}{4}$$
$$P(A \cap B) = \frac{1}{8}$$

$$(ii) P(B \cap A') = P(B) - P(B \cap A)$$
$$P(B \cap A') = P(B) - P(A \cap B)$$
$$P(B \cap A') = \frac{3}{8} - \frac{1}{8}$$
$$P(B \cap A') = \frac{1}{4}$$

$$(iii) P(A' \cup B') = P(A \cap B)' = 1 - P(A \cap B)$$
$$P(A' \cup B') = 1 - \frac{1}{8} = \frac{7}{8}$$

$$(iv) P(A' \cap B') = P(A \cup B) = 1 - P(A \cap B)$$
$$P(A' \cap B') = 1 - \frac{3}{4} = \frac{1}{4}$$

$$(v) P(B' \cap A) = P(A) - P(A \cap B)$$
$$P(B' \cap A) = \frac{1}{2} - \frac{1}{8}$$
$$P(B' \cap A) = \frac{3}{8}$$

[4] A problem in Statistics is given to three students A, B, and C whose chances of solving the same are respectively. If all the three students solve the problem independently, what is the probability that the problem will be solved?

Solution: The problem will be solved if at least one of the three students solves the problem.

Let, A be the events that the student solve the problem.

B be the events that the student solve the problem.

C be the events that the student solve the problem.

$P(A)=\frac{1}{2}, P(B)=\frac{1}{3}$ and $P(C)=\frac{1}{4}$ $\quad P(A\cup B\cup C)=?$

Assuming that A, B and C are mutual independence, we have

$P(A\cap B)=P(A)\times P(B) \quad \Rightarrow P(A\cap B)=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$

$P(A\cap C)=P(A)\times P(C) \quad \Rightarrow P(A\cap C)=\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$

$P(B\cap C)=P(B)\times P(C) \quad \Rightarrow P(B\cap C)=\frac{1}{3}\times\frac{1}{4}=\frac{1}{12}$

$P(A\cap B\cap C)=P(A)\times P(B)\times P(C)$

$P(A\cap B\cap C)=\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}=\frac{1}{24}$

$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)$
$-P(B\cap C)+P(A\cap B\cap C)$

$P(A\cup B\cup C)=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\frac{1}{12}+\frac{1}{24}$

$P(A\cup B\cup C)=\frac{12+8+6-4-3-2+1}{24}$

$P(A\cup B\cup C)=\frac{3}{4}$

[5] A problem in mathematics is given to five students A, B,C, D and E. Their chances of solving it are, $\frac{1}{2},\frac{1}{3},\frac{1}{3},\frac{1}{4}$ and $\frac{1}{5}$ respectively. Find the probability that the problem will

(a) not be solved

(b) be solved

Solution:

Let, A be the events that the student solve the problem.

B be the events that the student solve the problem.

C be the events that the student solve the problem.

D be the events that the student solve the problem.

E be the events that the student solve the problem.

$$P(A)=\frac{1}{2},P(B)=\frac{1}{3},P(C)=\frac{1}{3},P(D)=\frac{1}{4}\text{ and }P(E)=\frac{1}{5}$$

(a) The problem will not be solved when none of the students solve it. So the required probability is:

$$P(\text{Problem will not be solved})=P(A\cap B\cap C\cap D\cap E)$$

$$P(A\cap B\cap C\cap D\cap E)=P(A')\times P(B')\times P(C')\times P(D')\times P(E')$$

$$P(A'\cap B'\cap C'\cap D'\cap E')=\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)$$

$$P(A'\cap B'\cap C'\cap D'\cap E')=\frac{1}{2}\times\frac{2}{3}\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}$$

$$P(A'\cap B'\cap C'\cap D'\cap E')=\frac{2}{15}$$

(b) The problem will be solved when at least one of the students solve it. So the required probability is:

$$P(\text{Problem will be solved})=P(A\cup B\cup C\cup D\cup E)=1-P(A\cup B\cup C\cup D\cup E)'$$

$$P(A\cup B\cup C\cup D\cup E)=1-\frac{2}{15}$$

$$P(A\cup B\cup C\cup D\cup E)=\frac{13}{15}$$

[6] A bag contains 5 balls & it is not known how many of them are white 2 balls are drawn at random from the bag & they are noted to be white. What is the chance that all the balls in the bag are white?

Solution :- Since 2 white balls have been drawn out the bag must have contained 2,3,4 ,or 5 white balls

Let B_1 be the event of bag containing 2 white balls

B_2 be the event of bag containing 3 white balls

B_3 be the event of bag containing 4 white balls

B_4 be the event of bag containing 5 white balls

Let A be the event of drawing 2 white balls

$$P(A / B_1) = \frac{{}^2C_2}{{}^5C_2} = \frac{1}{10}$$

$$P(A / B_2) = \frac{{}^3C_2}{{}^5C_2} = \frac{3}{10}$$

$$P(A / B_3) = \frac{{}^4C_2}{{}^5C_2} = \frac{3}{5}$$

$$P(A / B_4) = \frac{{}^5C_2}{{}^5C_2} = 1$$

Since, the number of white balls in the bag is not known B_i are equally likely

$$P(B_1) = P(B_2) = P(B_3) = P(B_4) = \frac{1}{4}$$

Bayes theorem,

$$P(B_4/A) = \frac{P(B_4) \times P(A/B_4)}{\sum_{i=1}^{4} P(B_i) \times P(A/B_i)}$$

$$P(B_4/A) = \frac{\frac{1}{4}\times 1}{\frac{1}{4}\times\left(\frac{1}{10}+\frac{3}{10}+\frac{3}{5}+1\right)}$$

$$P(B_4/A) = \frac{1}{2}$$

[7] There are 3 true coins & 1 false coins with head on both sides a coins is chosen at random & tossed 4 times .If head occur all the 4 times what is the probability that the false coin has been chosen and use ?

Solution:- P(T)= P(the coin is a true coin) $= \frac{3}{4} - \frac{{}^3c_1}{{}^4c_1}$

P(F)= P(the coin is false coin)$= \frac{1}{4} = \frac{{}^1c_1}{{}^4c_1}$

Let A = Event of getting all heads in 4 tosses .Then

$$P(A / T) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16} \text{ and } \quad P(A / F) = 1$$

By Bayes theorem,

$$P(F/A) = \frac{P(F) \times P(A/F)}{P(F) \times P(A/F) + P(T) \times P(A/T)}$$

$$P(F/A) = \frac{\frac{1}{4}\times 1}{\left(\frac{1}{4}\times 1 + \frac{3}{4}\times 1/16\right)} = \frac{16}{19}$$

[8] Three identical urns A, B, C contains respectively. 2 whites, 3 black balls; 4 whites, 5 black balls; 3 whites, 4 black balls. One ball is drawn at random from any one of A, B, C. If the ball is white, find the probability that it is drawn from the urn C.

Solution:-

2w	4w	3w
3b	5b	3b
A	B	C

Let the events B_1, B_2, B_3 are the events that the chosen urns A, B, C respectively & the event A is the ball drawn is ‘white’. Since the urns are identical.

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3} \text{ \& } P(A/B_1) = \frac{2}{5}, P(A/B_2) = \frac{4}{9}, P(A/B_3) = \frac{3}{7}$$

$$\sum P(B_i) \times P(A/B_i) = P(B_1) \times P(A/B_1) + P(B_2) \times P(A/B_2) + P(B_3) \times P(A/B_3)$$

$$\sum P(B_i) \times P(A/B_i) = \frac{1}{3}\times\frac{2}{5} + \frac{1}{3}\times\frac{4}{9} + \frac{1}{3}\times\frac{3}{7} = \frac{401}{945}$$

Probability that the white ball is drawn from the urn $C = P(B_3/A)$

$$\frac{P(B_3)\times P(A/B_3)}{\sum_{i=1}^{n} P(B_i)\times P(A/B_i)} = \frac{\frac{1}{3}\times\frac{3}{7}}{\frac{401}{945}}$$

$$\frac{P(B_3)\times P(A/B_3)}{\sum_{i=1}^{n} P(B_i)\times P(A/B_i)} = \frac{1}{7}\times\frac{945}{401}$$

$$P(B_3/A)=\frac{135}{401}=0.3366$$

[9] Two urns A and B contain 6 white and 4 black balls; 5 white and 7 black balls res. If two balls are transferred from A to B & finally one black ball is drawn from the 2nd urn B. Find the probability that the balls transferred from the box A was 1 white & 1 black

Solution:-

Let us assume the following events.

B_1 be the transferred balls are both white

B_2 be the transferred balls are 1 white & 1 black

B_3 be the transferred balls are both black

A be the ultimately ball drawn from the urn B is black

It is required to find the probability $P\left(B_2/A\right)$

6w	5w
4b	7b

Here,

$$P(B_1)=\frac{^{6}C_2}{^{10}C_2}=\frac{15}{45}=\frac{1}{3}$$

$$P(B_2)=\frac{^{6}C_1\times {^{4}C_1}}{^{10}c_2}=\frac{24}{45}=\frac{8}{15}$$

$$P(B_3)=\frac{^{4}C_2}{^{10}C_2}=\frac{6}{45}=\frac{2}{15}$$

Again,

$P(A/B_1)$ = probability that the ultimate draw from the urn ‘B’ is black assuming the transferred balls are both white $= \frac{7}{14} = \frac{1}{2}$

$P(A/B_2)$ = probability that the ultimate draw from the urn ‘B’ is black assuming the transferred balls are one white and one black = $\frac{8}{14} = \frac{4}{7}$

$P(A/B_3)$= probability that the ultimate draw from the urn ‘B’ is black assuming the transferred balls are both black = $\frac{9}{14}$

So, by Bayes’ theorem, we have

$$P(B_2/A) = \frac{P(B_2) \times P(A/B_2)}{P(B_1) \times P(A/B_1) + P(B_2) \times P(A/B_2) + P(B_3) \times P(A/B_3)}$$

$$P(B_2/A) = \frac{\frac{8}{5} \times \frac{4}{7}}{\frac{1}{3} \times \frac{1}{2} + \frac{8}{15} \times \frac{4}{7} + \frac{2}{15} \times \frac{9}{14}}$$

$$\therefore P(B_2/A) = \frac{64}{117}$$

[10] Suppose that a product is produced in three factories X, Y and Z. It is known that the factory X produces thrice as many items as factory Y and that factories Y and Z produce the same number of items. Assume that it is known that three percent of item produced by each of the factories X and Y are defective while 5 percent of those manufactured by factory Y are defective. All the items produced in the three factories are stocked and an item of product is selected at random.

(i) What is the probability that this item is defective?

(ii) If an item selected at random is found to be defective, what is the probability that it was produced by factory X, Y and Z respectively?

Solution:- Let the number of items produced by each of the factories Y and Z be n. Then the number of items produced by the factory X is 3n.

Let E_1, E_2 and E_3 denotes the event that the items are produced by factory X, Y and Z respectively and Let A be the event of the item being defective. Then we have,

$$P(E_1) = \frac{3n}{3n+n+n} = 0.6$$

$$P(E_2) = \frac{n}{5n} = 0.2$$

$$P(E_3) = \frac{n}{5n} = 0.2$$

Also,

$$P\left(\frac{A}{E_1}\right) = P\left(\frac{A}{E_3}\right) = 0.03 \text{ and } P\left(\frac{A}{E_2}\right) = 0.05 \text{ (given)}$$

i] The probability that an item selected at random from the stock is defective is given by:

$$P(A) = \sum_{i=1}^{3} P(A \cap E_i) = \sum_{i=1}^{3} P(E_i) P\left(\frac{A}{E_i}\right)$$

$$P(A) = P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + P(E_3)P\left(\frac{A}{E_3}\right)$$

$$P(A) = 0.6 \times 0.03 + 0.2 \times 0.05 + 0.2 \times 0.03$$

$$P(A) = 0.034$$

(ii) By Bayes rule, the required probabilities are given by:

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(A)} = \frac{0.6 \times 0.3}{0.034} = \frac{9}{17}$$

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)P\left(\frac{A}{E_2}\right)}{P(A)} = \frac{0.2 \times 0.05}{0.034} = \frac{0.010}{0.034} = \frac{5}{17}$$

$$P\left(\frac{E_3}{A}\right) = \frac{P(E_3)P\left(\frac{A}{E_3}\right)}{P(A)} = \frac{0.006}{0.034} = \frac{3}{17}$$

$$P\left(\frac{E_3}{A}\right) = 1 - \left[P\left(\frac{E_1}{A}\right) + P\left(\frac{E_2}{A}\right)\right]$$

$$P\left(\frac{E_3}{A}\right) = 1 - \left(\frac{9}{17} + \frac{5}{17}\right)$$

$$P\left(\frac{E_3}{A}\right) = \frac{3}{17}$$

[11] In 2002 there will be three candidates for the position of principal- Mr. Chatterji, Mr. Ayangar and Dr. Singh whose chance of getting the appointment are in the proportion 4:2:3 respectively . The probability that Mr. Chatterji if selected would introduce co-education in the college is 0.3 .The probability of Mr. Ayangar and Dr. Singh doing the same are respectively 0.5 and 0.8

(i) What is probability that there will be co-education in the college in 2003?

(ii) If there is co-education in the college in 2003What is the probability that Dr. Singh is the principal.

Solution:-

Let us define the following event:

A: Introduction of co-education

E_1: Mr. Chatterji is selected as Principal

E_2: Mr. Ayangar is selected as a Principal

E_3: Dr. Singh is selected as Principal

Then we are given;

$$P(E_1) = \frac{4}{9},\ P(E_2) = \frac{2}{9}\ \&\ P(E_3) = \frac{3}{9}$$

$$P\left(\frac{A}{E_1}\right) = \frac{3}{10},\ P\left(\frac{A}{E_2}\right) = \frac{5}{10}\ \&\ P\left(\frac{A}{E_3}\right) = \frac{8}{10}$$

(i) The required probability that there will be co-education in the college in 2003 is given by:

$$P(A)=\left[P(A\cap E_1)\cup P(A\cap E_2)\cup P(A\cap E_3)\right]$$
$$P(A)=\left[P(A\cap E_1)+P(A\cap E_2)+P(A\cap E_3)\right]$$
$$P(A)=P(E_1)P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)+P(E_3)P\left(\frac{A}{E_3}\right)$$
$$P(A)=\frac{4}{9}\times\frac{3}{10}+\frac{2}{9}\times\frac{5}{10}+\frac{3}{9}\times\frac{8}{10}$$
$$P(A)=\frac{46}{90}$$
$$P(A)=\frac{23}{45}$$

[12] A company uses a 'selling aptitude test' in the selection of salesman past experience has shown that only 70% of all person applying for a sales position achieved classification "dissatisfactory" in actual, selling whereas reminder were classified as "satisfactory" 85% had scored a passing grade on the aptitude test only 25% of their classified unsatisfactory, had passed the test on the Basis of this information. What is the probability that a candidate would be a satisfactory salesman given that he passed the aptitude test?

Solution:- If s stands for a satisfactory classification as a salesman and p stand for passing the test then the probability that a candidate would be satisfactory salesman given that he passed aptitude test is

$$P(S / P) = \frac{(0.70)\ (0.85)}{(0.70)\ (0.85) + (0.30)\ (0.25)}$$
$$P(S / P) = \frac{0.595}{0.595 + 0.075}$$
$$P(S / P) = 0.888$$

The result indicates that the test is of value in screening candidate. Assuming no change in the type of candidate Applying for the selling positions the probability that a random applicant would be satisfactory is

70% on the other hand, if the company only accepts an applicant if the passed the test, probability increases to 0.888

[13] Consider three urn's containing white (W), black(B) & red(R) balls as follows:

URN I: 2W, 3B and 4R balls

URN II: 3W, 1B and 2R balls

URN III: 4W, 2B and 5R balls

Two balls are drawn from a urn and they happen to be one white and one Red ball. Find the probability that the balls are drawn from urn III

Solution:- Let E_1, E_2, E_3 be a three events that the urn I, II, III selected

$$\therefore P(E_1) = \frac{1}{3}, P(E_2) = \frac{1}{3} \text{ and } P(E_3) = \frac{1}{3}$$

A be the event that out of two selected balls, one is white and other is red.

Now, probability selecting 1W and 1R balls from urn is

$$\therefore P(A/E_1) = \frac{{}^2C_1 \times {}^4C_1}{{}^9C_2} = \frac{2\times4}{\frac{9!}{2!\times7!}} = \frac{2\times4}{\frac{9\times8}{2\times1}} = \frac{2\times4}{9\times4} = \frac{2}{9}$$

Similarly

$$\therefore P(A/E_2) = \frac{{}^3C_1 \times {}^2C_1}{{}^6C_2} = \frac{3\times2}{\frac{6!}{2!\times4!}} = \frac{3\times2}{\frac{6\times5}{2\times1}} = \frac{3\times2}{3\times5} = \frac{2}{5}$$

$$\therefore P(A/E_3) = \frac{{}^4C_1 \times {}^5C_1}{{}^{11}C_2} = \frac{4\times5}{\frac{11!}{2!\times9!}} = \frac{4\times5}{\frac{11\times10}{2\times1}} = \frac{4\times5}{11\times5} = \frac{4}{11}$$

Now, Bayes probability that ball drawn be long to URN III is

$$\therefore P(E_3/A) = \frac{{}^2C_1 \times {}^4C_1}{{}^9C_2} = \frac{2\times4}{\frac{9!}{2!\times7!}} = \frac{2\times4}{\frac{9\times8}{2\times1}} = \frac{3\times2}{3\times5} = \frac{2}{5}$$

$$P(E_3/A)=\frac{P(A/E_3)\ P(E_3)}{\sum_{i=1}^{3} P(A/E_i)\ (A/E_i)}$$

$$P(E_3/A)=\frac{P(A/E_3)\ P(E_3)}{P(A/E_1)\ P(E_1) + P(A/E_2)\ P(E_2) + P(A/E_3)\ P(E_3)}$$

$$P(E_3/A)=\frac{\frac{4}{11}\times\frac{1}{3}}{\frac{2}{9}\times\frac{1}{3}+\frac{2}{5}\times\frac{1}{3}+\frac{4}{11}\times\frac{1}{3}}=\frac{\frac{4}{11}\times\frac{1}{3}}{\frac{1}{3}\left(\frac{2}{3}+\frac{2}{5}+\frac{4}{11}\right)}$$

$$P(E_3/A)=\frac{\frac{4}{11}}{\frac{110+198+180}{495}}=\frac{\frac{4}{11}}{\frac{488}{495}}=\frac{4}{11}\times\frac{495}{488}=\frac{45}{122}$$

(ii) The required probability is given by Bayes' rule by

$$P\left(\frac{E_3}{A}\right)=\frac{P(E_3)P\left(\frac{A}{E_3}\right)}{P(A)}$$

$$P\left(\frac{E_3}{A}\right)=\frac{\frac{3}{9}\times\frac{8}{10}}{\frac{46}{90}}$$

$$P\left(\frac{E_3}{A}\right)=\frac{24}{46}$$

$$P\left(\frac{E_3}{A}\right)=\frac{12}{23}$$

[14] The probability of X, Y and Z becoming managers are $\frac{4}{9},\frac{2}{9}$ and $\frac{1}{3}$ respectively. The probabilities that the bonus scheme will be introduced if X, Y and Z becomes manager are $\frac{3}{10},\frac{1}{2}$ and $\frac{4}{5}$ respectively.

(i) What is the probability that bonus scheme will be introduced and

(ii) If the bonus scheme has introduced, what is the probability that the manager appointed was X?

Solution: Let, E_1, E_2, E_3 denote the event that X, Y and Z become managers respectively and A denotes event that 'Bonus Scheme' is introduced.

We are given:

$A = (E_1 \cap A) \cup (E_2 \cap A) \cup (E_3 \cap A)$

$P(E_1) = \frac{4}{9},\ P(E_2) = \frac{2}{9}, P(E_3) = \frac{1}{3}$

$P(A/E_1) = \frac{3}{10}, P(A/E_2) = \frac{1}{2}, P(A/E_3) = \frac{4}{5}$

The event A can materialize in the following mutually exclusive ways:

(i) Mr. X becomes manager and bonus scheme is introduced i.e.

(i) $E_1 \cap A$ happens

(ii) $E_2 \cap A$ happens

(iii) $E_3 \cap A$ happens

Then,

Thus, $A = \left(E_1 \cap A\right) \cup \left(E_2 \cap A\right) \cup \left(E_3 \cap A\right)$

Where,

$\left(E_1 \cap A\right), \left(E_2 \cap A\right)$ and $P(E_1/A) = 0.20$ are disjoint.

(i) Using Addition theorem of probability, the required probability that the bonus scheme is introduced is given by:

$P\left(A\right) = \left(E_1 \cap A\right) + \left(E_2 \cap A\right) + \left(E_3 \cap A\right)$

$P(A) = P(E_1) \times P(A / E_1) + P(E_2) \times P(A / E_2) + P(E_3) \times P(A / E_3)$

$P(A) = \frac{4}{9} \times \frac{3}{10} + \frac{2}{9} \times \frac{1}{2} + \frac{1}{3} \times \frac{4}{5}$

$P(A) = \frac{23}{45}$

$P(A) = 0.51$

(ii) Using Bayes' theorem of the required probability is:

$$P(E_1/A)=\frac{P(E_1)\times P(A/E_1)}{P(A)}$$

$$P(E_1/A) = \frac{12/90}{23/45}$$

$$P(E_1/A) = \frac{6}{23}$$

$$P(E_1/A) = 0.20$$

[15] A factory produces a certain type of outputs by three type of machine. The respective daily production figures are:

Machine -I: 3000 unites, machines-II: 2500 units, machine-III: 4500 units.

Past experience shows that 1 percent of the output produced by machine I is defective, the corresponding fraction of defective for the other two machines are 1.2% and 2% resp. An item is drawn at random from the day's production run and is found to be defective. What is probability that it comes from the output of

(i) Machine-I, (ii) Machine-II and (iii) Machine-III

Answer:-

Let E_1, E_2 and E_3 denotes the event that the output is produced by machine I, II and III. And let A denotes the event that the output is defective and then we have:

$$P(E_1) = \frac{3000}{10000} = 0.30$$

$$P(E_3) = \frac{4500}{10000} = 0.45$$

$P(A/E_1) = 1\% = 0.01$,

$P(A/E_2) = 1.2\% = 0.012$

$P(A/E_3) = 2\% = 0.02$

The probability that an item selected at random from days production is defective is given by

$$P(A)=\sum_{i=1}^{3} P(E_i \cap A)$$

$$P(A)=\sum P(E_i)\times P(A / E_i)$$

$$P(A)=0.30\times 0.01+0.25\times 0.012+0.42\times 0.02$$

$$P(A)=0.015$$

By Bayes rule the required probabilities given by ;

(i) $P(E_1/A)=\dfrac{P(E_1)\times P(A/E_1)}{P(A)}=\dfrac{0.003}{0.015}=\dfrac{1}{5}$

(ii) $P(E_2/A)=\dfrac{P(E_2)\times P(A/E_2)}{P(A)}=\dfrac{0.003}{0.015}=\dfrac{1}{5}$

(iii) $P(E_3/A)=\dfrac{P(E_3)\times P(A/E_3)}{P(A)}=\dfrac{0.009}{0.015}=\dfrac{3}{5}$

The probabilities in (i), (ii) and (iii) are known as posterior probabilities of events E_1, $E_{2,}$ and E_3 respectively.

[16] An Economist believes that during periods of high economic growth, the Indian Rupee appreciates with probability 0.70; in periods of moderate economic growth, it appreciates with probability 0.40; and during periods of low economic growth, the Rupee appreciates with probability 0.20.During any period of time the probability of high economic growth is 0.30; the probability of moderate economic growth is 0.50 and the probability of low economic growth is 0.20. Suppose the Rupee value has been appreciating during the present period. What is the probability that we are experiencing the period of (a) high, (b) moderate, and (c) low, economic growth?

Solution: Our partition consists of three events: high economic growth (event H), moderate economic growth (event M) and low economic growth (event L). The prior probabilities of these events are:

P(H) = 0.30; P(M) = 0.50; P(L) = 0.20

Let A be the event that the rupee appreciates. We have the conditional probabilities

$P(A/H) = 0.70;\quad P(A/M) = 0.40;\quad P(A/L) = 0.20$

By using the Bayes' theorem we can find out the required probabilities

$P(H/A);\quad P(M/A);\quad P(L/A)$

Now,

$$P(H/A) = \frac{P(A/H) \times P(H)}{P(A/H) \times P(H) + P(A/M) \times P(M) + P(A/L) \times P(L)}$$

$$P(H/A) = \frac{0.70 \times 0.30}{0.70 \times 0.30 + 0.40 \times 0.50 + 0.20 \times 0.20} = 0.467$$

$$P(M/A) = \frac{P(A/M) \times P(M)}{P(A/H) \times P(H) + P(A/M) \times P(M) + P(A/L) \times P(L)}$$

$$P(H/A) = \frac{0.40 \times 0.50}{0.70 \times 0.30 + 0.40 \times 0.50 + 0.20 \times 0.20} = 0.444$$

$$P(L/A) = \frac{P(A/L) \times P(L)}{P(A/H) \times P(H) + P(A/M) \times P(M) + P(A/L) \times P(L)}$$

$$P(H/A) = \frac{0.70 \times 0.30}{0.70 \times 0.30 + 0.40 \times 0.50 + 0.20 \times 0.20} = 0.089$$

Multiple Choice Questions (MCQ)

Choose the correct alternative from the following:

[1] If P(A) = P(B), then the two events A and B are

(a) Independent (b) Dependent

(c) Equally likely (d) Both (a) and (c).

Answer: (c) Equally likely

[2] If for two events A and B, $P(A \cap B) \neq P(A) \times P(B)$), then the two events A and B are

(a) Independent (b) Dependent

(c) Not equally likely (d) Not exhaustive

Answer: (b) Dependent

[3] If P(A/B) = P(A), then

(a) A is independent of B

(b) B is independent of A

(c) B is dependent of A

(d) Both (a) and (b)

Answer: (d) Both (a) and (b)

[4] If two events A and B are independent, then

(a) A and the complement of B are independent

(b) B and the complement of A are independent

(c) Complements of A and B are independent

(d) All of these (a), (b) and (c)

Answer: (d) All of these (a), (b) and (c)

[5] If two events A and B are independent, then

(a) They can be mutually exclusive

(b) They can not be mutually exclusive

(d) They cannot be exhaustive

(d) Both (b) and (c)

Answer: (b) They cannot be mutually exclusive

[6] If two events A and B are mutually exclusive, then

(a) They are always independent

(b) They may be independent

(c) They cannot be independent

(d) They cannot be equally likely

Answer: (c) They cannot be independent

[7] If a coin is tossed twice, then the events 'occurrence of one head, 'occurrence of two heads and 'occurrence of no head' are

(a) Independent

(b) Equally likely

(c) Not equally likely

(d) Both (a) and (b)

Answer: (c) Not equally likely

[8] P (B/A) is defined only when

(a) A is a sure event

(b) B is a sure event

(c) A is not an impossible event

(d) B is an impossible event

Answer: (c) A is not an impossible event

[9] P (A/B') is defined only when

(a) B is not a sure event

(b) B is a sure event

(c) B is an impossible event

(d) B is not an impossible event

Answer: (a) B is not a sure event

[10] For two events A and B, $P(A \cup B) = P(A) + P(B)$ only when

(a) A and Bare equally likely events

(b) A and B are exhaustive events

(c) A and B are mutually independent

(d) A and B are mutually exclusive

Answer: (d) A and B are mutually exclusive

[11] The number of conditions to be satisfied by three events A, B and C for complete independence is

(a) 2

(b) 3

(c) 4

(d) NOTA

Answer: (c) 4

[12] Which of the following statement is true?

(a) A and A’ form partition of Ω

(b) A and Ω form partition of Ω

(c) A and A’ do not form partition of Ω

(d) Only two events cannot form partition of Ω

Answer: (a) A and A’ form partition of Ω

Self –Assessment Questions:

[1] Explain the concept of independence of two events.

[2] Define independence of two events A and B on Ω. Give an illustration.

[3] Does independence of two events imply that the events are mutually exclusive? Justify your answer.

[4] Does mutually exclusiveness of events imply independence? Justify?

[5] Given A and B two independent events defined on a sample space Ω, prove that

(i) A and B' are independent

(ii) A' and B are independent.

(iii) A' and B'are independent

[6] Given three events A, B, C defined on a sample space Ω, define

(i) Pairwise independence

(ii) Mutual independence of A, B, C.

[7] Prove or disprove the following:

(i) Mutual independence of three events implies that Pairwise independence.

(ii) Pairwise independence implies that mutual independence.

[8] Explain the concept of conditional probability.

[9] For two events defined on a sample space Ω. Define conditional probabilities P (A/B) and P (B/A).

[10] Show that conditional probability satisfies all the axioms of probability.

[11] In a certain school, examination results showed that 20% students failed in Mathematics, 5% failed in English while 10% failed in both Mathematics and English. Are the two events 'failing in Mathematics' and 'failing in English' independent?

[12] An article manufactured by a company consists of two parts A and B. In the manufacturing process of part A, 9 out of 100 are likely to be defective, similarly 5 out of 100 are likely to be defective in the process of part B. Calculate the probability that the assembled parts will not be defective.

[13] A husband and wife appear for two vacancies in the same post. The probability of husband's selection is 1/7 and that of wife's selection is 1/5. What is the probability that (i) both of them will be selected? (ii) only one of them will be selected ? (iii) none of them will be selected?

Previous Year Questions in University Exam:

[1] Define independence of two events A and B on Ω. (**April-2015**)

[2] If A and B are independent events with P(A) = 0.4 and P(B) = 0.5, then P (AUB) is…. (**April-2015**)

(A) 0.7 (B) 0.6 (C) 0.9 (D) 0.8

[3] If P(A ∩ B) = 0, then the two events A and B are (**April-2016**)

(A) Exhaustive Events (B) Dependent Events

(C) Mutually Exclusive Events (D) NOTA

[4] Define conditional probability of an event. (**April-2015**)

[5] Give that $P(A_1)=P(A_2) = P(A_3)=\frac{1}{3}$ and $P(B/A_1)=\frac{2}{7}$, $P(B/A_2)=\frac{4}{9}$

$P(B/A_3)=\frac{1}{5}$, find $P(A_1/B)$. (**April-2015, April-2017**)

[6] State and prove Bayes' theorem. (**April-2015, Octo-2015, April-2016**)

[7] If A, B, C are any three events defined on Ω, with P (B) > 0, then prove that $P(A \cup C/B)=P(A/B)+P(C/B)-P(A \cap C/B)$. (**April-2015**)

[8] Define complete independence of three events defined on a sample space Ω.

(**April-2015**)

[9] If A and B are independent events with P(A) = 0.3, P(B) = 0.5, find $P(A' \cap B')$. (**Octo-2015**)

[10] If A and B are independent events with P(A) = 0.6, P(B) = 0.5, find $P(A \cap B)$. (**April-2016**)

[11] State the multiplication theorem for two sample space Ω.

(**Octo-2015), April-2016, April-2017.**

[12] If A and B are mutually exclusive and non-empty events defined on sample space Ω. Show that $P(A/A \cup B)=\frac{P(A)}{P(A)+P(A)}$. (**April-2016**)

[13] Define the following terms : (i) Pairwise independence of three events. (ii) Partition of the sample space. (**April-2016**)

[14] Define: conditional probability of an event. (**Octo-2016, April-2017**)